Exact Results on the Antiferromagnetic Three-State Potts Model

Wang, Swendsen, and Kotecký[1] recently presented high-precision numerical data from Monte Carlo simulations of antiferromagnetic three-state Potts models. In two dimensions on a square lattice this model has a zero-temperature critical point related to the six-vertex model at the ice point. We exploit this relation to calculate exact values of the critical exponent γ/ν and the amplitude of finite-size corrections to the free energy. Our results agree well with the numerical values obtained by Wang, Swendsen, and Kotecký.[1]

At $T=0$ the antiferromagnetic Potts model finds a ground state in which each spin $S(\mathbf{r})\equiv\exp[i2\pi\sigma(\mathbf{r})/3]$ $(\sigma=0,1,2)$ takes on a value different from any of its nearest neighbors. This is the well known three coloring problem, which can be solved exactly[2] by mapping onto a six-vertex model on the dual lattice. Each six-vertex configuration corresponds to three distinct Potts configurations. The partition function of the Potts model is thus 3 times the partition function of an associated six-vertex model.

Care must be taken in relating the boundary conditions of these models. Consider systems of size $L_x\times L_y$ where, for simplicity, we consider only even values of L_x and L_y. Imposing periodic boundary conditions on the Potts spins limits the possible six-vertex configurations to those with polarizations of the form $P_x=3m$ and $P_y=3n$ (m and n are integers). Polarizations are defined by subtracting the number of down-pointing (or left-pointing) arrows from the number of up-pointing (or right-pointing) arrows and dividing by 2. We will also need *step* boundary conditions, where $S(\mathbf{r})=S(\mathbf{r}+L_x\hat{\mathbf{x}})\times\exp(\pm i2\pi/3)$. In this case the allowed polarizations are $P_x=3m\pm1$ and $P_y=3n$.

The leading finite-size corrections of a six-vertex model partition function with fixed polarizations are related to those of the Gaussian model partition function with *step* boundary conditions[3,4] so that

$$Z_{\text{Potts}}\propto 3\sum_{P_x,P_y} Z_g^{(P_x,P_y)}\,, \tag{1}$$

with $Z_g^{(P_x,P_y)}$ the partition function of the Gaussian model with $\phi(\mathbf{r}+L_k\hat{\mathbf{e}}_k)=\phi(\mathbf{r})+P_k$ $(k=x,y)$. The Gaussian coupling constant K_g takes the value of $2\pi/3$ at the ice point.[3] From this expression we extract the limiting behavior as $L_x,L_y\to\infty$ (e.g., see Ref. 4). The free energy takes the form

$$f(L_x,L_y)=f_{\text{bulk}}-\frac{1}{L_xL_y}\ln\left[\frac{\theta_3(3s)\theta_3(s/3)}{\eta^2(s)}\right], \tag{2}$$

where θ_3 is a Jacobi θ function of the third kind and η is the Dedekind η function (see Ref. 4 for a definition of θ_3 and η). $s=L_y/L_x$ is the aspect ratio of the lattice. The calculations of Wang, Swendsen, and Kotecký[1] were carried out on lattices with $s=1$, for which we predict finite-size corrections of the form A/L_x^2 with $A=\ln 2.93577965\ldots$. This value of A fits the numerical data very well.

Wang, Swendsen, and Kotecký[1] also obtained a value of the critical exponent $\gamma/\nu\simeq1.666(2)$ from the size dependence of the staggered magnetization. We find it is exactly $\frac{5}{3}$. The idea is to force an interface into the Potts spin system by employing *step* boundary conditions. Repeating the previous calculation with the new boundary we find an excess free energy per unit length for the interface, $\zeta(s)$. In the infinite-cylinder limit, $s\to\infty$, the interfacial free energy becomes[5]

$$\zeta(s=\infty)=2\pi x_s/L_x\,. \tag{3}$$

We identify $x_s=\frac{1}{6}$ as the staggered magnetization critical exponent. This value of x_s can also be obtained from the work of den Nijs, Nightingale, and Schick.[6] We then obtain the critical exponent for staggered susceptibility from the elementary finite-size-scaling theory, $\gamma/\nu=2-2x_s=5/3$.

In conclusion, we have obtained exact results at the critical point of the three-state antiferromagnetic Potts model. These results can be easily generalized to include next-nearest-neighbor interactions. The $T=0$ Potts model is then related to the general zero-field six-vertex model and results similar to Eqs. (2) and (3) follow. We note that measurement of finite-size-scaling amplitudes by Monte Carlo simulations can be useful in determining the nature of critical points, and also that knowledge of the finite-size-scaling amplitudes can be used to speed up convergence of numerical data.[7]

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