Does Hard Core Interaction Change Absorbing-Type Critical Phenomena?

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It has been generally believed that hard core interaction is irrelevant to absorbing-type critical phenomena because the particle density is so low near an absorbing phase transition. We study the effect of hard core interaction on the N-species branching annihilating random walks with two offspring and report that hard core interaction drastically changes the absorbing-type critical phenomena in a nontrivial way. Through a Langevin equation-type approach, we predict analytically the values of the scaling exponents, $\nu_\perp = 2$, $z = 2$, $\alpha = 1/2$, and $\beta = 2$ in one dimension for all $N > 1$. Direct numerical simulations confirm our prediction. When the diffusion coefficients for different species are not identical, $\nu_\perp$ and β vary continuously with the ratios between the coefficients.

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The study of nonequilibrium systems with trapped (absorbing) states has been very active in recent years [1]. Models displaying absorbing phase transitions describe a wide range of phenomena, for example, epidemic spreading, catalytic chemical reactions, surface growth, wetting and roughening, self-organized criticality, and transport in disordered media [1,2]. Furthermore, the absorbing transition is one of the simplest and natural extensions of the well-established equilibrium phase transition to nonequilibrium systems, which are still poorly understood.

The concept of universality, which plays a key role in equilibrium critical phenomena, was shown to be applicable also to nonequilibrium absorbing transitions. Critical behavior near an absorbing transition is determined by properties such as dimensionality and symmetry and is not affected by details of the system. Finding a new universality class is difficult, and only a few classes of absorbing transitions are known [1].

Hard core interaction between particles or kinks has been believed to be irrelevant to absorbing-type critical phenomena, because the particle density is so low near an absorbing transition that the probability of multiple occupations at a site should be too small to be significant. This conventional belief leads to recent successes of field theoretical techniques using bosonic-type operators [3–7]. However, it is well known that hard core interaction *does* change the asymptotic decay behavior of the particle density in many multi-species diffusion-reaction models near an annihilation fixed point [8]. Since many absorbing transition models can be mapped onto diffusion-reaction ones, it may seem natural to ask a question whether hard core constraint changes the absorbing-type universality classes in multi-species models. Despite recent efforts using fermionic formulation incorporating hard core interactions [9,10], the effect of hard core interactions is barely understood both analytically and numerically.

In this Letter, we study the N-species branching annihilating random walks with two offspring [N-BAW(2)], introduced recently by Cardy and Täuber [7]. The model was solved exactly for all $N > 1$, using renormalization group techniques in bosonic-type formulation which ignores hard core interactions. We employ a Langevin equation-type approach incorporating hard core interactions and predict analytically the values of critical exponents associated with the absorbing transition. It turns out that the hard core interaction drastically changes the universality class in a nontrivial way and the critical exponents vary continuously with the ratio of diffusion constants of different species. Our predictions may not be exact due to the approximate nature of the Langevin equation approach, but direct numerical simulations confirm our predictions.

The N-BAW(2) model is a classical stochastic system consisting of N species of particles, A_i $(i = 1, \ldots, N)$. Each particle diffuses on a d-dimensional lattice with two competing dynamic processes: pair annihilation and branching. Pair annihilation is allowed only between identical particles ($A_i + A_i \rightarrow \emptyset$). In the branching process, a particle A_i creates two identical particles in its neighborhood ($A_i \rightarrow A_i + 2A_j$), with rate σ for $i = j$ and rate $\sigma'/(N - 1)$ for $i \neq j$.

For $N = 1$, this model exhibits an absorbing transition of directed Ising-type (Z_2 symmetry) at a finite branching rate [11–14]. The N-species generalization imposes the permutation symmetry S_N between species. Like in the Potts-type generalization of the absorbing transition models [15], this model for $N > 1$ is always active except at the annihilation fixed point of zero branching rate.

Critical properties near the annihilation fixed point have been explored exactly by Cardy and Täuber for $N > 1$ in the framework of bosonic field theory [7]. The upper critical dimension d_c is 2. Using a perturbation expansion, they showed that the branching process associated with σ is irrelevant. For $\sigma = 0$, it was found that the models for all $N > 1$ are active for $\sigma' \neq 0$ and their scaling behavior near the annihilation fixed point ($\sigma' = \sigma'_c = 0$) forms a

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new universality class independent of N. For $d < d_c$, the critical behavior is characterized by the exponents

$$\nu_\perp = 1/d, \qquad z = 2, \quad \alpha = d/2, \quad \beta = 1. \tag{1}$$

Here, the exponents are defined as

$$\begin{aligned} \xi &\sim \Delta^{-\nu_\perp}, \qquad \tau \sim \xi^z, \\ \rho(t) &\sim t^{-\alpha}, \qquad \rho_s \sim \Delta^\beta, \end{aligned} \tag{2}$$

where $\Delta = \sigma' - \sigma'_c$, ξ is the correlation length, τ the characteristic time, $\rho(t)$ the particle density at time t, and ρ_s the steady-state particle density.

Even in the presence of a hard core interaction, the scaling exponents α and z should follow from the simple random walk exponents; $z = 2$ and $\alpha = d/z$ for $d < d_c$ [4]. Near the annihilation fixed point, elementary scaling theory ensures $\beta = \nu_\perp z \alpha$, which leads to $\beta = \nu_\perp d$. We determine the value of $\nu_\perp$ through a Langevin equation-type approach.

The particle density can change by branching processes and pair annihilation processes. If we start with a configuration of very low particle density, the particle density initially grows by branching processes, $(A_i \rightarrow A_i + 2A_j)$. In this growth regime, a newly created pair of offspring and its parent are far more likely to annihilate against each other than other particles in the system. Dynamics of such three particle configurations or "triplet" governs the growth behavior of the particle density and the inter-triplet interactions can be ignored. The particle density growth will be finally capped by pair annihilations processes of independent particles and the system reaches a steady state.

We focus on the growth regime dominated by triplet dynamics, from which $\nu_\perp$ can be evaluated [7]. We consider only the case of $\sigma = 0$, where a newly created pair is always dissimilar to its parent. The survival probability $S(t)$ of the triplet of the same species decays much faster $(\sim t^{-3/2})$ than that of different species, so the branching process associated with σ is irrelevant. Near $\sigma' = 0$, the time evolution of the particle density of the ith species, $\rho_i(t)$, is written as

$$\frac{d\rho_i}{dt} = 2\frac{\sigma'}{N-1}\sum_{j\neq i}\left[\rho_j(t) - \int_0^t L_{ij}(t-t')\rho_j(t')dt'\right], \tag{3}$$

where $L_{ij}(t - t')dt$ is the probability that an ith-species pair created by a jth-species particle at time t', annihilates in an interval between t and $t + dt$. The two terms in the right-hand side represent the creation and annihilation process of a triplet, respectively. Pair annihilation contribution from independent particles is $\mathcal{O}(\rho^2)$, which is ignored in the growth regime [16].

The kernel $L_{ij}(t)$ is simply related to the survival probability $S_{ij}(t)$ of the triplet $(A_j + 2A_i)$ as $L_{ij}(t) = -dS_{ij}(t)/dt$. To keep the lowest order of σ' in Eq. (3), we evaluate $S_{ij}(t)$ at $\sigma' = 0$. When hard core interaction is not present, a pair of A_i's does not see its parent A_j, so annihilate each other freely by random walks. In that case, it is well known that $S(t) = S_{ij}(t)$ decays asymptotically as $S(t) \sim t^{-\delta}$ with $\delta = 1 - d/2$ for $d < 2$ and becomes finite ($\delta = 0$) for $d > 2$, irrespective of their diffusion constants [17]. However, with hard core interaction, the pair annihilation process changes significantly due to an effective bias in the diffusive behavior, generated by the parent particle A_j. The motion of A_i near A_j picks up a convective component with velocity proportional to $t^{-1/2}$, so the convective displacement is of the same order of diffusive displacement $t^{1/2}$. In this case, the competition between the convection and diffusion becomes nontrivial and the scaling exponent δ depends continuously on the parameters of the system [18].

We calculate the survival probability $S(t)$ of a triplet in one dimension. With hard core interaction, $S(t)$ depends crucially on where to create two offspring with respect to their parent. When two offspring are divided by their parent (static branching) [13], they have no chance to meet each other. The survival probability never decays ($\delta = 0$). When two offspring are placed both to the left side or both to the right side of the parent particle with equal probability (dynamic branching) [13], $S(t)$ decays with a nontrivial scaling exponent.

Consider three random walkers on a line, labeled as A, B, and C. A is a parent particle that created two offspring, B and C, to the right side of A. Two offspring B and C are of the same species, which is different from their parent A. Hard core repulsion is present between A and B. B and C annihilate instantaneously upon collision. The calculation of $S(t)$ belongs to the class of problems known as the "capture process" [17–19].

Let the coordinates of the walkers be x_A, x_B, and x_C and their diffusion coefficients D_A, D_B, and D_C, respectively. In our case, $D_B = D_C$. It is useful to introduce the scaled coordinates $y_i = x_i/\sqrt{D_i}$, where $i = A, B, C$. Then we can map this triplet system to a single walker system with isotropic diffusion in three-dimensional space (y_A, y_B, y_C) [17,20]. The walker survives inside the wedge bounded by two planes: a *reflecting* plane P_r of $\sqrt{D_A}\, y_A = \sqrt{D_B}\, y_B$ and an *absorbing* plane P_a of $\sqrt{D_B}\, y_B = \sqrt{D_C}\, y_C$.

The survival probability $S(t)$ of an isotropic random walker in a d-dimensional cone with absorbing boundary is known [21]. In particular, $S(t)$ in a wedge with an opening angle Θ asymptotically decays as $t^{-\pi/2\Theta}$ [21]. In our case, one of the boundary planes, P_r, is not absorbing but reflecting. The probability of the walker at P_r is nonzero and there is no net flux across this plane. Using this fact, one can easily show that our system should be equivalent to the system in a wedge bounded by two absorbing planes with twice the opening angle.

We find that the survival probability of the triplet decays with the exponent

$$\delta = \frac{\pi}{4\Theta} = \left[\frac{4}{\pi}\cos^{-1}\left(\frac{1}{\sqrt{2(1+r)}}\right)\right]^{-1}, \tag{4}$$

where Θ is the opening angle of the wedge and $r = D_A/D_B$. The exponent δ monotonically decreases from 1 to 1/2 as the diffusivity ratio r varies from 0 to ∞. At $r = 1$ (the same diffusivity for all walkers), $\delta = 3/4$.

First, we consider the case that diffusion coefficients are identical for all species. The N-coupled Langevin equations, Eq. (3), can be simplified in terms of the total particle density, $\rho(t) = \sum_i \rho_i(t)$, as

$$\frac{d\rho}{dt} = 2\sigma'\rho(t) - 2\sigma' \int_0^t L(t - t')\rho(t')\,dt', \quad (5)$$

where $L(t) = L_{ij}(t)$ is independent of i and j. Taking Laplace transformation, we find

$$s\tilde{\rho}(s) - \rho(0) = 2\sigma'[1 - \tilde{L}(s)]\,\tilde{\rho}(s) = 2\sigma' s\tilde{S}(s)\tilde{\rho}(s), \quad (6)$$

where $\tilde{\rho}(s) = \int_0^\infty \rho(t)e^{-st}\,dt$, and similarly $\tilde{L}(s)$ and $\tilde{S}(s)$ are the Laplace transform of $L(t)$ and $S(t)$, respectively. With $S(t) \sim t^{-\delta}$, one can show that $\tilde{S}(s) \sim s^{\delta-1}$ for $\delta > 0$.

The function $\tilde{\rho}(s)$ has a pole in the positive real axis at $s_0 \sim \sigma'^{1/(1-\delta)}$. When the initial density $\rho(0)$ is small, the density $\rho(t)$ increases exponentially as $\exp(s_0 t)$. Using the definition of the characteristic time τ [Eq. (2)], we find

$$\tau = \sigma'^{-\nu_\perp z} = 1/s_o = \sigma'^{-1/(1-\delta)}. \quad (7)$$

With $\delta = 3/4$ for the dynamic branching model, we arrive at $\nu_\perp z = 1/(1 - \delta) = 4$. Therefore we predict that the critical exponents for the dynamic branching N-BAW(2) model with hard core interaction in one dimension are

$$\nu_\perp = 2, \qquad z = 2, \qquad \alpha = 1/2, \qquad \beta = 2, \quad (8)$$

which should be valid for all $N > 1$. For the static branching N-BAW(2) model, $\delta = 0$ and $\nu_\perp = \beta = 1/2$. Without hard core interactions, branching methods do not matter and $\delta = 1/2$, which leads to Eq. (1).

We check the above predictions for the N-BAW(2) model by direct numerical simulations for $N = 2$, 3, and 4. We start with a pair of particles. With probability p, a randomly chosen particle (A_i) creates two offspring ($2A_j$) on two nearest neighboring sites (dynamic/static branching). The branching probability p is distributed as γp for $i = j$ and $(1 - \gamma)p/(N - 1)$ for $i \neq j$. Otherwise, the particle hops to a nearest neighboring site. Two particles of the same species at a site annihilate instantaneously. In cases of models with hard core interactions, branching/hopping attempts are rejected when two particles of different species try to occupy the same site. Critical probability $p_c = 0$ for all models considered here.

We measure the total particle density ρ_s in the steady state, averaged over $5 \times 10^2 \sim 5 \times 10^4$ independent samples for several values of $\Delta = p - p_c$ (0.001 $\sim$ 0.05) and lattice size L ($2^5 \sim 2^{11}$). We set $\gamma = 1/2$. Using the finite-size scaling theory [22]

$$\rho_s(\Delta, L) = L^{-\beta/\nu_\perp} F(\Delta L^{1/\nu_\perp}), \quad (9)$$

the value of $\nu_\perp$ is determined by "collapsing" data of ρ_s with $\beta/\nu_\perp = 1$ (Fig. 1). Numerical data show that $\nu_\perp$ does not depend on N in all models as expected. We find $\nu_\perp = 1.00(5)$ for models without hard core interactions, which agrees with the result by Cardy and Täuber [7]. With hard core interactions, we find $\nu_\perp = 1.9(1)$ for the dynamic branching models and $\nu_\perp = 0.50(3)$ for the static branching models, which confirm our predictions within statistical errors.

When the diffusion coefficients are not identical for different species, $S_{ij}(t)$ decays with the exponent δ depending on diffusivity ratio $r = D_j/D_i$. Instead of a single Langevin equation, we are then forced to deal with the N-coupled Langevin equations. The solution of the system of equations is difficult in general, but the equations become quite simple for $N = 2$.

Laplace-transformed coupled equations for $N = 2$ become

$$\begin{aligned} s\tilde{\rho}_1(s) - \rho_1(0) &= 2\sigma' s\tilde{S}_{12}(s)\tilde{\rho}_2(s), \\ s\tilde{\rho}_2(s) - \rho_2(0) &= 2\sigma' s\tilde{S}_{21}(s)\tilde{\rho}_1(s). \end{aligned} \quad (10)$$

We take $\rho_2(0) = 0$ as an initial condition and solve the equations for $\tilde{\rho}_1(s)$:

$$s\tilde{\rho}_1(s) - \rho_1(0) = 4\sigma'^2 s\tilde{S}_{12}\tilde{S}_{21}\tilde{\rho}_1(s). \quad (11)$$

Note that $S_{12}(t)$ decays with exponent $\delta(r)$ with $r = D_2/D_1$ and $S_{21}(t)$ with $\delta(1/r)$; see Eq. (4). From the pole position of $\tilde{\rho}_1(s)$, we arrive at

$$\nu_\perp(r) = \frac{1}{2 - \delta(r) - \delta(1/r)}. \quad (12)$$

The exponent $\delta(r)$ ranges from 1/2 to 1, but $\delta(r) + \delta(1/r)$ varies only slightly with r. It ranges from 3/2 to 1.5255, so $\nu_\perp(r)$ varies within only a few percent. Because of rather large statistical errors ($\sim$10%), we could not confirm numerically the r dependence of $\nu_\perp$. However, it is clear from our derivation that $\nu_\perp$ should vary continuously with diffusivity ratio. Although we were not able to obtain a similar expression for $\nu_\perp$ for general N, we expect that $\nu_\perp$ varies continuously but only slightly with r for all $N > 1$.

In summary, we showed that hard core interaction in the N-BAW(2) model changes its universality class in a nontrivial way. Details of branching methods (static/dynamic branching) and also the diffusivity ratios between different species change drastically the absorbing-type critical phenomena. We find that, for all $N > 1$, the dynamic branching models with hard core interaction form a new universality class, different from the models without hard core interaction. Especially, the scaling exponents vary continuously with the diffusivity ratios. The static branching models with hard core interaction form yet another new universality class. Therefore, one should not trust the conventional bosonic formalism in studying multi-species

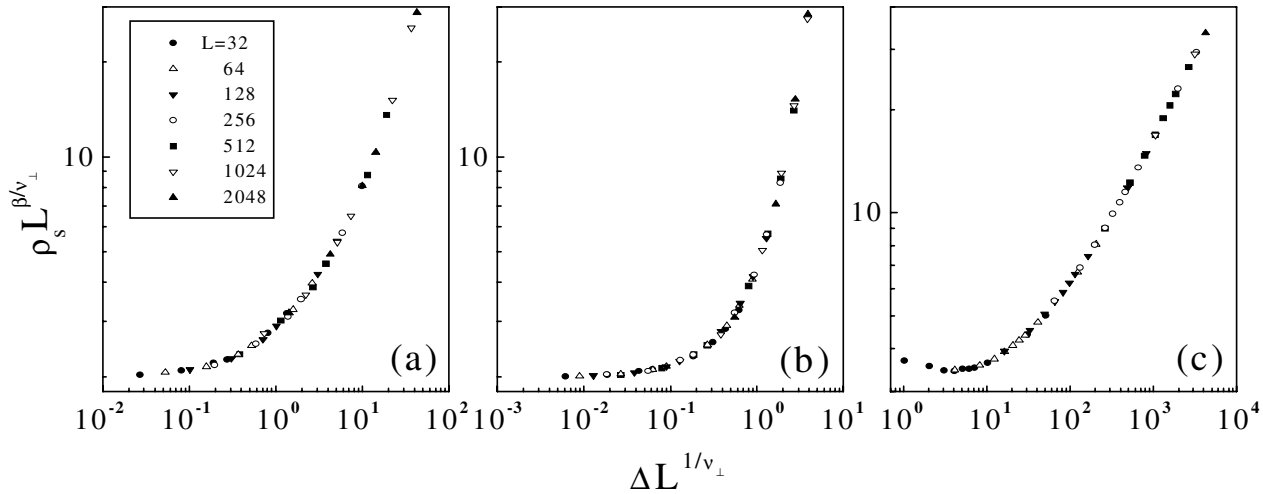

FIG. 1. Data collapse of $\rho_s L^{\beta/\nu_\perp}$ against $\Delta L^{1/\nu_\perp}$ with $\beta/\nu_\perp = 1$ for various system sizes $L = 2^5, \ldots, 2^{11}$ for $N = 2$ BAW(2) models (a) without hard core interaction, (b) with hard core interaction (dynamic branching), and (c) with hard core interaction (static branching). The best collapses are achieved with (a) $\nu_\perp = 1.00(5)$, (b) 1.9(1), and (c) 0.50(3), respectively.

models in general. Numerical simulations confirm most of our predictions, but large scale simulations are necessary to measure the diffusivity ratio dependence of the scaling exponents.

The present analytic method to study the effect of hard core interaction can be applied to a wide range of multi-species diffusion-reaction models near the annihilation fixed point. Our analysis implies that many multi-species models with hard core interaction may exhibit a nontrivial absorbing phase transition with continuously varying exponents.

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