Fluctuation Theorems and Entropy Production with Odd-Parity Variables

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We show that the total entropy production in stochastic processes with odd-parity variables (under time reversal) is separated into three parts, only two of which satisfy the integral fluctuation theorems in general. One is the usual excess contribution that can appear only transiently and is called nonadiabatic. Another one is attributed solely to the breakage of detailed balance. The last part that does not satisfy the fluctuation theorem comes from the steady-state distribution asymmetry for odd-parity variables that is activated in a nontransient manner. The latter two parts combine together as the housekeeping (adiabatic) contribution, whose positivity is not guaranteed except when the excess contribution completely vanishes. Our finding reveals that the equilibrium requires the steady-state distribution symmetry for odd-parity variables independently, in addition to the usual detailed balance.

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The (integral) fluctuation theorem (FT) [1–5] can be stated for a variable $\mathcal{R}_r$ (or $\mathcal{R}$, in brief) assigned to a random sequence of states (or event) r [6] as

$$\langle e^{-\mathcal{R}} \rangle \equiv \sum_r \mathcal{P}_r e^{-\mathcal{R}_r} = 1, \tag{1}$$

where $\mathcal{P}_r$ is the probability of a sequence r. As a corollary, Jensen's inequality guarantees $\langle \mathcal{R} \rangle \geq 0$. Consider r as a path or trajectory in state space, generated during a time interval by a stochastic dynamics. In the case when its functional $\mathcal{R}$ [7] represents the total entropy production during the process, the FT has been derived for various nonequilibrium (NEQ) processes, and the thermodynamic second law $\langle \Delta S_{\mathrm{tot}} \rangle \geq 0$ automatically follows Refs. [3,4,8].

Hatano and Sasa found that a part of the total entropy production (excess contribution), ΔS_{ex} [9], also satisfies the FT, which represents the entropy production associated with transitions between steady states [10,11]. Later, Speck and Seifert showed that the remaining part (the housekeeping contribution), ΔS_{hk} [9], also satisfies the FT, which is indispensable to maintain the NEQ steady state (NESS) [12,13]. In the case of (quasistatic) reversible processes, the system almost always stays at equilibrium during the process; then, the housekeeping contribution vanishes, $\Delta S_{\mathrm{hk}}^{\mathrm{eq}} = 0$. Recently, Esposito and Van den Broeck [6] interpreted the housekeeping contribution as an adiabatic (nontransient) part and the excess contribution as a nonadiabatic (transient) part of the total entropy production, through a time scale argument.

Most findings about the FTs including the above separation of the total entropy production have only been obtained when all state variables have even parity under time reversal, such as position variables. Including odd-parity variables, such as momentum or spin, the mathematical description becomes more complicated, in particular, for NEQ processes. A typical example is a driven Brownian motion in the underdamped case with a non-potential-type or momentum-dependent driving force. [14,15].

Very recently, Spinney and Ford suggested a separation of the total entropy production into three terms for stochastic systems with odd-parity variables [16]. The excess part ΔS_{ex} can be separated out, which satisfies the FT. The housekeeping part ΔS_{hk} is composed of two distinct terms, and only one term (ΔS_2) satisfies the FT. More surprisingly, the other term (ΔS_3) that does not satisfy the FT turns out to be transient, which seems inconsistent with the usual adiabatic feature of the housekeeping contribution and rather shares the similar relaxation feature with ΔS_{ex} [17]. Based on this observation, it was argued that the existing classification of the total entropy production (adiabatic vs nonadiabatic) does not hold, and thus the separation line between the excess and the housekeeping parts is blurred with odd-parity variables.

In this Letter, we show that the existing classification is still valid with odd-parity variables and that there are clear-cut separation lines between entropy productions with distinct physical origins. It is crucial to recognize that the equilibrium (reversible) processes require not only the detailed balance (DB) relation but also the symmetry of the steady-state distribution (SSD) for odd-parity variables, which turn out to be two independent constraints. Violation of either one brings about an independent nonvanishing housekeeping contribution, and the processes become irreversible even in the steady state.

We introduce a natural and unique splitting scheme of the housekeeping contribution into two fundamentally different nontransient parts. The first part, ΔS_{bDB}, represents the DB breakage exclusively, and the second part, ΔS_{as}, originates from the SSD asymmetry for odd-parity variables. $\Delta S_{\rm as}$ disappears without odd-parity variables. Each part is nontransient, and thus the total housekeeping contribution maintains its adiabatic feature. The total entropy production is divided into the housekeeping and excess contributions as usual, respectively. The excess part $\Delta S_{\rm ex}$ is the same as in the even-variable-only case, which satisfies the FT, as does the total entropy $\Delta S_{\rm tot} = \Delta S_{\rm hk} + \Delta S_{\rm ex}$. In contrast, neither $\Delta S_{\rm as}$ nor $\Delta S_{\rm hk} = \Delta S_{\rm bDB} + \Delta S_{\rm as}$ obeys the FT, while $\Delta S_{\rm bDB}$ does.

A stochastic process can be described by the master equation

$$\dot{p}_x(t) = \sum_y \omega_{x,y}[\lambda(t)] p_y(t), \tag{2}$$

where $p_x(t)$ is the probability distribution of state x at time t and $\omega_{x,y}$ is the transition rate from y to x for $x \neq y$ with $\omega_{y,y} = -\sum_{x\neq y}\omega_{x,y}(<0)$. x represents a state vector $(s_1, s_2, \ldots)$ where each component s_k represents a state variable with a definite parity, $\epsilon_k = 1$ (even) or $\epsilon_k = -1$ (odd), under time reversal. The time-reversed state is given by $\epsilon x = (\epsilon_1 s_1, \epsilon_2 s_2, \ldots)$. $\lambda(t)$ denotes a time-dependent protocol as a set of external control parameters.

Figure 1 shows a path $\mathbf{x}(t)$ generated by the master equation with the transition rate matrix $\omega = \{\omega_{x,y}\}$ from $t = 0$ to τ, and its time-reversed path $\tilde{\mathbf{x}}(t)$ is defined as $\epsilon\mathbf{x}(\tau - t)$. We assume that there are N jumping processes between different states at times $\{t_1, \ldots, t_N\}$. Then, the probability functional of the "forward" path $\mathbf{x}(t)$ reads

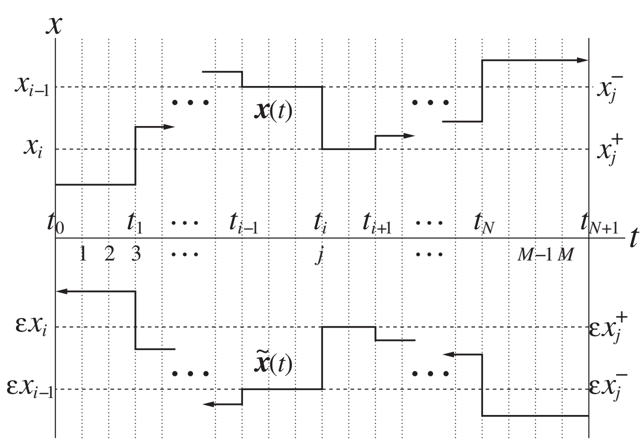

FIG. 1. Schematic of a sample path $\mathbf{x}(t)$ and its time-reversed path $\tilde{\mathbf{x}}(\tau - t)$. The horizontal axis represents time t with $t_0 = 0$ and $t_{N+1} = \tau$. The vertical axis represents state x in the upper half of the figure and the time-reversed state ϵx in the lower half. There are two time indices. Index i is used for N jumping processes between different states at times $\{t_1, \ldots, t_N\}$. j is used for the time-discretized version such as $t = j\Delta t$ ($j = 1, \ldots, M$), with $\tau = (M+1)\Delta t$ in the $\Delta t \to 0$ limit. Note that x_i is the state that is kept unchanged during a time interval from t_i to t_{i+1}.

$$\mathcal{P}_\omega[\mathbf{x}] \propto p_{x_0}\left(\prod_{i=0}^{N-1} e^{\int_{t_i}^{t_{i+1}} dt \omega_{x_i,x_i}[\lambda(t)]} \omega_{x_{i+1},x_i}(\lambda_{i+1})\right) \times e^{\int_{t_N}^{\tau} dt \omega_{x_N,x_N}[\lambda(t)]}, \tag{3}$$

where p_{x_0} is the probability distribution of initial state x_0, x_i is the state for $t_i < t < t_{i+1}$, and $\lambda_i = \lambda(t_i)$. The time-reversed process is considered under the protocol changes of $\lambda(t) \to \lambda(\tau - t)$, and the initial probability is chosen as the final probability of the forward process, p_{x_N}. After a proper rearrangement (see Refs. [6,16] for details), the probability functional of the "reverse" path $\tilde{\mathbf{x}}(t)$ reads

$$\mathcal{P}_\omega[\tilde{\mathbf{x}}] \propto p_{x_N}\left(\prod_{i=0}^{N-1} e^{\int_{t_i}^{t_{i+1}} dt \omega_{\epsilon x_i,\epsilon x_i}[\lambda(t)]} \omega_{\epsilon x_i,\epsilon x_{i+1}}(\lambda_{i+1})\right) \times e^{\int_{t_N}^{\tau} dt \omega_{\epsilon x_N,\epsilon x_N}[\lambda(t)]}. \tag{4}$$

We remark that Eqs. (3) and (4) have the same normalization factor since both include the same number of jumping processes.

The path-dependent total entropy production, $\Delta S_{\rm tot}[\mathbf{x}]$, is the measure of the irreversibility of a path $\mathbf{x}$ with respect to its time-reversed path $\tilde{\mathbf{x}}$, which can be defined as the associated path probability ratio [12,13]

$$\Delta S_{\rm tot}[\mathbf{x}] = \ln\frac{\mathcal{P}_\omega[\mathbf{x}]}{\mathcal{P}_\omega[\tilde{\mathbf{x}}]}. \tag{5}$$

Note that $\Delta S_{\rm tot}$ is a FT functional since it satisfies Eq. (1); $\langle e^{-\Delta S_{\rm tot}}\rangle = \sum_{\mathbf{x}} \mathcal{P}_\omega[\mathbf{x}] e^{-\Delta S_{\rm tot}} = \sum_{\tilde{\mathbf{x}}} \mathcal{P}_\omega[\tilde{\mathbf{x}}] = 1$ (Jacobian $|\partial\tilde{\mathbf{x}}/\partial\mathbf{x}| = 1$). If there are only even-parity variables (all $\epsilon_k = 1$), the exponential factors of staying probabilities in Eqs. (3) and (4) are identical. These factors are completely canceled out in the probability ratio, and thus only transition rates matter in $\Delta S_{\rm tot}$. However, it does not work in that way when odd-parity variables are included, and this is a main source of mathematical difficulty and also of different physical origins.

It is convenient to express the path probability by the conditional probability for transition from y to x during discretized unit time Δt (Fig. 1), given as

$$\Gamma_{x,y}(\lambda(t)) = \delta_{x,y} + \omega_{x,y}[\lambda(t)]\Delta t, \tag{6}$$

where $\delta_{x,y}$ is the Kronecker delta valued 1 for $x = y$ and 0 otherwise. Δt is chosen small enough to maintain $\Gamma_{x,x} > 0$. Then, the two path probabilities can be written as

$$\begin{aligned} \mathcal{P}_\Gamma[\mathbf{x}] &= p_{x_0}\prod_{j=1}^{M}\Gamma_{x_j^+,x_j^-}(\lambda_j), \\ \mathcal{P}_\Gamma[\tilde{\mathbf{x}}] &= p_{x_N}\prod_{j=1}^{M}\Gamma_{\epsilon x_j^-,\epsilon x_j^+}(\lambda_j), \end{aligned} \tag{7}$$

where x_j^+ and x_j^- represent states just after and before time $t = j\Delta t$, respectively, and $\lambda_j = \lambda(j\Delta t)$. Note that the product therein includes the staying processes of $x_j^+ = x_j^-$ as well as the jumping processes. Using Eq. (7), one simply writes $\Delta S_{\rm tot}$ as

$$\Delta S_{\rm tot} = \Delta S + \sum_{j=1}^{M} \ln \frac{\Gamma_{x_j^+, x_j^-}(\lambda_j)}{\Gamma_{\epsilon x_j^-, \epsilon x_j^+}(\lambda_j)}, \tag{8}$$

where $\Delta S = -\ln(p_{x_N}/p_{x_0})$ is the entropy change of the system for the forward path. We will later take the $\Delta t \to 0$ limit to come back to the original problem. The explicit path dependence of the entropy production is dropped just for simplicity.

The breakage of the DB is an essential characteristic of nonequilibrium processes that leads to entropy production even in the NESS. Thus, it would be useful to search for a separation scheme to isolate the entropy production due to the DB breakage only. The generalized (instantaneous) DB condition at time t for stochastic processes with odd-parity variables is given as $\omega_{x,y}[\lambda(t)]p_y^s[\lambda(t)] = \omega_{\epsilon y,\epsilon x}[\lambda(t)]p_{\epsilon x}^s[\lambda(t)]$ for $x \neq y$, where $p_x^s[\lambda(t)]$ is the SSD of state x for a constant protocol λ, whose value is given by $\lambda(t)$, satisfying the steady-state equation $\sum_x \omega_{y,x}(\lambda)p_x^s(\lambda) = 0$. This condition guarantees no physical average currents between states in the steady state and also yields a relation regarding the diagonal elements as $\omega_{x,x}[\lambda(t)]p_x^s[\lambda(t)] = \omega_{\epsilon x,\epsilon x}[\lambda(t)]p_{\epsilon x}^s[\lambda(t)]$, using $\omega_{x,x} = -\sum_{y\neq x}\omega_{y,x}$. In terms of the conditional probabilities, the generalized DB condition thus reads as

$$\Gamma_{x,y}(\lambda(t)) = \Gamma_{\epsilon y,\epsilon x}(\lambda(t)) \frac{p_{\epsilon x}^s[\lambda(t)]}{p_y^s[\lambda(t)]} + \left[1 - \frac{p_{\epsilon x}^s[\lambda(t)]}{p_x^s[\lambda(t)]}\right]\delta_{x,y},$$
$$= \delta_{x,y} + \omega_{\epsilon y,\epsilon x}[\lambda(t)] \frac{p_{\epsilon x}^s[\lambda(t)]}{p_y^s[\lambda(t)]}\Delta t. \tag{9}$$

We propose the adjoint stochastic process with $\Gamma_{x,y}^\dagger$ that can be used to provide a precise measure of the broken DB as

$$\Gamma_{x,y}^\dagger(\lambda(t)) = \delta_{x,y} + \omega_{x,y}^\dagger[\lambda(t)]\Delta t, \tag{10}$$

with

$$\omega_{x,y}^\dagger = \omega_{\epsilon y,\epsilon x}\frac{p_{\epsilon x}^s}{p_y^s}. \tag{11}$$

It is trivial to show that $\Gamma^\dagger$ is stochastic with sufficiently small Δt [19]: $\sum_x \Gamma_{x,y}^\dagger = 1$ and $\Gamma_{x,y}^\dagger \geq 0$ for all x, y.

When $\Gamma_{x,y}^\dagger = \Gamma_{x,y}$, the DB is satisfied. The entropy production due to the DB breakage, $\Delta S_{\rm dDB}$, can be defined as

$$\Delta S_{\rm bDB} = \sum_{j=1}^{M} \ln \frac{\Gamma_{x_j^+, x_j^-}(\lambda_j)}{\Gamma^\dagger_{x_j^+, x_j^-}(\lambda_j)} = \ln \frac{\mathcal{P}_\Gamma[\mathbf{x}]}{\mathcal{P}_{\Gamma^\dagger}[\mathbf{x}]}, \tag{12}$$

where $\mathcal{P}_{\Gamma^\dagger}[\mathbf{x}] = p_{x_0}\prod_{j=1}^M \Gamma^\dagger_{x_j^+,x_j^-}(\lambda_j)$ is the probability of the forward path $\mathbf{x}$ by the adjoint dynamics. $\Delta S_{\rm bDB}$ is a FT functional by itself, satisfying the integral FT, and must belong to the housekeeping contribution since it contributes even in the steady state. It also satisfies the detailed FT: $P(R)/P^\dagger(-R) = e^R$, where $P(R)$ is the probability that $\Delta S_{\rm bDB} = R$ in the original process while $P^\dagger$ is its counterpart in the adjoint process. This is because the mapping to the adjoint dynamics is involutive ($\Gamma^{\dagger\dagger} = \Gamma$) [6] since both the original and adjoint dynamics share the same SSD ($p_x^s = p_x^{\dagger s}$).

Now, subtracting $\Delta S_{\rm bDB}$ from $\Delta S_{\rm tot}$, one can write the remaining part, $\Delta S' = \Delta S_{\rm tot} - \Delta S_{\rm bDB}$ as

$$\Delta S' = \ln \frac{p_{x_0}}{p_{x_N}} + \sum_{j=1}^{M} \ln \frac{\Gamma^\dagger_{x_j^+, x_j^-}(\lambda_j)}{\Gamma_{\epsilon x_j^-, \epsilon x_j^+}(\lambda_j)}, \tag{13}$$

which is not a FT functional in general because it is not guaranteed to write down $\Delta S' = \ln \mathcal{P}_\Gamma[\mathbf{x}]/\mathcal{P}_{\Gamma'}[\mathbf{x}']$ for the probability functional $\mathcal{P}_{\Gamma'}[\mathbf{x}']$ of (reverse) path $\mathbf{x}'$ in a stochastic dynamics with a certain conditional probability Γ'. One can find the stochastic condition for $\Gamma'_{y,x} = \Gamma_{x,y}\Gamma_{\epsilon y,\epsilon x}/\Gamma^\dagger_{x,y}$ as

$$\sum_y \Gamma'_{y,x} = 1 + \left(\frac{p_x^s - p_{\epsilon x}^s}{p_{\epsilon x}^s}\right)\left(\frac{\Gamma^\dagger_{x,x} - \Gamma_{x,x}}{\Gamma^\dagger_{x,x}}\right). \tag{14}$$

This shows that Γ' is in general not stochastic due to ϵ mismatch (note that $\Gamma^\dagger_{x,x}$ also includes ϵ). Exceptions when $p_x^s = p_{\epsilon x}^s$ or $\Gamma^\dagger_{x,x} = \Gamma_{x,x}$ will be revisited later.

We can instead extract the excess contribution by introducing another stochastic process with $\Gamma^*_{x,y}$ (exactly the same one as in the even-variable-only case) as

$$\Gamma^*_{x,y}(\lambda(t)) = \delta_{x,y} + \omega^*_{x,y}[\lambda(t)]\Delta t, \tag{15}$$

with

$$\omega^*_{x,y} = \omega_{y,x}\frac{p_x^s}{p_y^s}. \tag{16}$$

Now, we define the excess contribution, $\Delta S_{\rm ex}$, as

$$\Delta S_{\rm ex} = \ln \frac{p_{x_0}}{p_{x_N}} + \sum_{j=1}^{M} \ln \frac{\Gamma_{x_j^+, x_j^-}(\lambda_j)}{\Gamma^*_{x_j^-, x_j^+}(\lambda_j)} = \ln \frac{\mathcal{P}_\Gamma[\mathbf{x}]}{\mathcal{P}_{\Gamma^*}[\hat{\mathbf{x}}]}$$
$$= \Delta S + \sum_{j=1}^{M} \ln \frac{p^s_{x_j^+}}{p^s_{x_j^-}}, \tag{17}$$

where the path for the Γ^* process is given by $\hat{\mathbf{x}}(t) = \mathbf{x}(\tau - t)$ (time reversed without parity change). Of course, $\Delta S_{\rm ex}$ is again a FT functional, satisfying the integral FT.

The remaining part, $\Delta S_{\rm as} = \Delta S_{\rm tot} - \Delta S_{\rm bDB} - \Delta S_{\rm ex}$, can be written as

$$\Delta S_{\rm as} = \sum_{j=1}^{M} \ln \left[\frac{p^s_{\epsilon x_j^+}}{p^s_{x_j^+}} + \delta_{x_j^+, x_j^-}\frac{p^s_{x_j^+} - p^s_{\epsilon x_j^+}}{p^s_{x_j^+}\Gamma_{\epsilon x_j^+, \epsilon x_j^+}}\right]. \tag{18}$$

One can easily show that this part does not satisfy the FT except by vanishing when there is a SSD symmetry as

$$p^s_{\epsilon x} = p^s_x \tag{19}$$

between mirror (opposite-parity) states. This asymmetric contribution $\Delta S_{\rm as}$ is present even in the absence of external driving $\lambda(t)$ and also in the NESS (clearly not transient), so it must belong to the housekeeping contribution. It therefore follows that

$$\Delta S_{\rm tot} = \Delta S_{\rm ex} + \Delta S_{\rm hk}, \tag{20}$$

with $\Delta S_{\rm hk} = \Delta S_{\rm bDB} + \Delta S_{\rm as}$, which does not obey the FT in general.

The total housekeeping contribution should vanish in the reversible (equilibrium) processes, which implies that the equilibrium condition requires not only the DB but also the symmetry between the SSD of the mirror states, when odd-parity variables are involved. These two conditions are independent, and our two housekeeping contributions, $\Delta S_{\rm bDB}$ and $\Delta S_{\rm as}$, measure precisely the violation of these two equilibrium conditions, respectively.

It is worthy of noting that $\Delta S_{\rm bDB}$ and $\Delta S_{\rm as}$ steadily contribute to $\Delta S_{\rm tot}$ in the adiabatic process (or even at $\dot\lambda = 0$), where the time scale of the $\lambda(t)$ change is much larger than the relaxation time. This time scale argument is the reasoning behind the classification of adiabatic and nonadiabatic contributions in $\Delta S_{\rm tot}$, proposed in Ref. [6]. In this criterion, both $\Delta S_{\rm bDB}$ and $\Delta S_{\rm as}$ are the adiabatic contributions while $\Delta S_{\rm ex}$ is the nonadiabatic one.

In the $\Delta t \to 0$ (i.e., $M \to \infty$) limit, one can obtain

$$\Delta S_{\rm ex} = \ln \frac{p_{x_0}}{p_{x_N}} + \sum_{i=1}^{N} \ln \frac{p^s_{x_i}[\lambda(t_i)]}{p^s_{x_{i-1}}[\lambda(t_i)]}, \tag{21}$$

$$\Delta S_{\rm hk} = \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} dt\{\omega_{x_i,x_i}[\lambda(t)] - \omega_{\epsilon x_i,\epsilon x_i}[\lambda(t)]\} + \sum_{i=1}^{N} \ln \frac{\omega_{x_i,x_{i-1}}[\lambda(t_i)] p^s_{x_{i-1}}[\lambda(t_i)]}{\omega_{\epsilon x_{i-1},\epsilon x_i}[\lambda(t_i)] p^s_{x_i}[\lambda(t_i)]}, \tag{22}$$

$$\Delta S_{\rm bDB} = \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} dt\{\omega_{x_i,x_i}[\lambda(t)] - \omega^\dagger_{x_i,x_i}[\lambda(t)]\} + \sum_{i=1}^{N} \ln \frac{\omega_{x_i,x_{i-1}}[\lambda(t_i)]}{\omega^\dagger_{x_i,x_{i-1}}[\lambda(t_i)]}, \tag{23}$$

$$\Delta S_{\rm as} = \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} dt \omega_{\epsilon x_i,\epsilon x_i}[\lambda(t)] \left(\frac{p^s_{\epsilon x_i}[\lambda(t)]}{p^s_{x_i}[\lambda(t)]} - 1\right) + \sum_{i=1}^{N} \ln \frac{p^s_{\epsilon x_i}[\lambda(t_i)]}{p^s_{x_i}[\lambda(t_i)]}. \tag{24}$$

$\Delta S_{\rm bDB}$ represents the contribution solely responsible for the DB breakage, which is the only housekeeping contribution in the absence of odd-parity variables. While a similar contribution was found by Spinney and Ford [16], their term (ΔS_2) contains what is not directly related to the broken DB. In the meantime, $\Delta S_{\rm as}$ is an odd-variable-specific term. It characterizes the asymmetry in the SSD for mirror states. Thus, the asymmetric contribution serves as another important quantity to measure the irreversibility of nonequilibrium processes. A similar term (ΔS_3) found by Spinney and Ford [16] only exists transiently and does not measure fully the asymmetric contribution.

It is not difficult to realize $\Delta S_{\rm as}$ in the underdamped case with a nonzero inertia term described by the second-order stochastic differential equations. For example, with an external torque on a rigid pendulum, a net angular motion is generated that breaks the symmetry of the SSD for mirror (opposite-angular-velocity) states [14]. A general non-potential-type force acting on a Brownian particle can produce a rotational or directed current that breaks the SSD symmetry. Explicit calculations on simple systems are currently under way [20].

In order to illustrate our results more explicitly, we consider a trivial one-particle model on a ring with L sites, where both the DB and SSD symmetry can be broken independently. A particle state is described by $x = (n, v)$, where n represents the particle position ($n = 1, \ldots, L$) and v is its velocity ($v = \pm 1$). The particle hops in the direction of its velocity with rate $\omega_{x,y} = h_v$ for $y = (n, v) \to x = (n + v, v)$. In addition, we allow velocity flips with rate $\omega_{x,y} = f_v$ for $y = (n, v) \to x = (n, -v)$. For simplicity, we assume h_v and f_v are constants, independent of n, and all other transitions are prohibited. This model is a simple generalization of the model discussed in Ref. [16], which is recovered with $h_v = h_{-v}$.

The stationary solutions are given by $p^s_x = f_{-v}/[L(f_v + f_{-v})]$ for $x = (n, v)$, which is symmetric for mirror states when $f_v = f_{-v}$. The DB condition is simply $h_v f_{-v} = h_{-v} f_v$, which guarantees no spatial particle current in the steady state. However, this is not enough to claim the time-reversal symmetry to maintain equilibrium due to the broken symmetry for time-reversed mirror states in general. Moreover, it is clear that these two conditions for equilibrium are independent each other.

One may derive the mean entropy production rates from Eqs. (12) and (18) as

$$\langle \dot S_{\rm bDB} \rangle = \sum_x p_x(t) \left[h_v \ln \frac{h_v f_{-v}}{h_{-v} f_v} + \frac{h_{-v} f_v - h_v f_{-v}}{f_{-v}} \right],$$
$$\langle \dot S_{\rm as} \rangle = \sum_x p_x(t) \left[(f_v - h_v) \ln \frac{f_{-v}}{f_v} + \frac{(f_{-v} - f_v)(f_{-v} + h_{-v})}{f_{-v}} \right], \tag{25}$$

with $x = (n, v)$. It is easy to check that $\langle \dot S_{\rm bDB} \rangle = 0$ with the DB condition and that $\langle \dot S_{\rm as} \rangle = 0$ with the SSD symmetry. We also find $\langle \dot S_{\rm bDB} \rangle \geq 0$ for any $p_x(t)$, as expected from its FT property, while $\langle \dot S_{\rm as} \rangle$ can take any value, depending on $p_x(t)$. Furthermore, in the NEQ steady state with $p_x(t) = p^s_x$, both mean entropy production rates become nonvanishing constants, which confirms the nontransient property of both entropy production contributions.

Finally, we briefly mention on the exceptional cases observed in Eq. (14), where the total entropy production can be divided into two terms, each of which satisfies the FT. Particulary, we consider the case in which $\Gamma^{\dagger}_{x,x} = \Gamma_{x,x}$ (the other case of $p^s_{\epsilon x} = p^s_x$ leads to the conventional separation by $\Delta S_{\rm as} = 0$). The condition gives a new stochastic process $\Gamma'_{y,x}$, distinct from $\Gamma^*_{y,x}$ in Eq. (15). Then, one readily finds a new separation as

$$\Delta S_{\rm tot} = \Delta S_{\rm bDB} + \Delta S_{\rm mix}, \tag{26}$$

where $\Delta S_{\rm mix} = \Delta S_{\rm ex} + \Delta S_{\rm as}$ also satisfies the FT. In the light of physical origin, $\Delta S_{\rm as}$ belongs to the adiabatic contribution. From the mathematical point of view, however, it operates with nonadiabatic $\Delta S_{\rm ex}$. Moreover, in the adiabatic limit, we have $\Delta S_{\rm hk}$ only, which can be cleanly separated into two FT functionals. It will be an interesting study to find an example of this exception.

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