# Fluctuation-Response Inequalities for Kinetic and Entropic Perturbations

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We derive fluctuation-response inequalities for Markov jump processes that link the fluctuations of general observables to the response to perturbations in the transition rates within a unified framework. These inequalities are derived using the Cramér-Rao bound, enabling broader applicability compared to existing fluctuation-response relations formulated for static responses of currentlike observables. The fluctuation-response inequalities are valid for a wider class of observables and are applicable to finite observation times through dynamic responses. Furthermore, we extend these inequalities to open quantum systems governed by the Lindblad quantum master equation and find the quantum fluctuation-response inequality, where dynamical activity plays a central role.

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*Introduction*—The response of a physical system to small perturbations is a fundamental aspect of its behavior [1] and is critical for understanding material properties such as conductivity [2] and viscoelasticity [3]. Near equilibrium, the seminal fluctuation-dissipation theorem states that the system's response is directly related to its spontaneous equilibrium fluctuations [4]. Much effort has been devoted to generalizing the connection between response and fluctuations in far-from-equilibrium regimes, expressed as equalities [5–12]. Although the recently discovered extended fluctuation-dissipation theorems relate the response function to a nonequilibrium correlation function, they often require detailed microscopic knowledge of the steady state or its dynamics [5–8,10]. In this context, complementary inequalities have been developed more recently to provide upper bounds on the response in terms of fluctuations [12–20].

Traditionally, response theory has primarily considered perturbations such as small impulses that alter a given potential [5–7]. In contrast, responses to changes in kinetic parameters—such as the mobility of a colloidal particle or the concentration of a catalyst in a chemical reaction—have been largely overlooked, as they affect only reaction rates without altering the equilibrium distribution, leading to vanishing responses at equilibrium [14,15,17]. However, under nonequilibrium conditions, such kinetic perturbations become crucial for fully capturing nonequilibrium responses. Recent studies have established explicit thermodynamic bounds on the kinetic responses of state-dependent observables across various processes [11,14–17] and on those of currentlike observables in Markov jump processes [12,19,20]. Among these, the response thermodynamic uncertainty relation (R-TUR) [19] states that the ratio of the kinetic response to the fluctuations of a currentlike observable is bounded by the entropy production (EP) rate. It was found that the R-TUR arises from an identity that connects kinetic response and fluctuations, valid even far from equilibrium [20]. Although the identities found in [20], coined fluctuation-response relations (FRRs), can also be used to derive other types of upper bounds on the response to perturbations in the symmetric and antisymmetric parts of transition rates, their validity is limited to infinitely long observation times.

In this Letter, we derive inequalities that relate the dynamic response to perturbations in transition rates with the fluctuations of a general observable in Markov jump processes, generalizing the FRRs by encompassing them as a limiting case. The derivation of these inequalities is based on the Cramér-Rao bound, a widely used tool for deriving various uncertainty relations [21–25]. An important advantage of this method is its simplicity and applicability to finite observation times in steady states, thereby extending the R-TUR for static response [19,20] to dynamic response. We further extend these inequalities to open quantum systems governed by the Lindblad quantum master equation.

*Setup*—We consider a continuous-time Markov jump process governed by the following master equation:

$$\dot{p}_i(t) = \sum_{j(\neq i)} [W_{ij} p_j(t) - W_{ji} p_i(t)], \tag{1}$$

where $W_{ij}$ denotes the transition rate from state $j$ to $i$, and $p_i(t)$ represents the probability of the system being in state $i$ at time $t$. We assume that every transition is bidirectional

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and that each pair of opposite transitions satisfies local detailed balance for thermodynamic consistency [26,27]. The transition rate is parametrized as [11,12,19]

$$W_{ij} = \exp\left(B_{ij}(\epsilon) + \frac{F_{ij}(\eta)}{2}\right), \tag{2}$$

where $B_{ij}$ and $F_{ij}$ represent the symmetric and antisymmetric parts of the transition rate, satisfying $B_{ij} = B_{ji}$ and $F_{ij} = -F_{ji}$, respectively. Intuitively, in the context of a reaction pathway, $B_{ij}$ represents the energy barrier between two states, while $F_{ij}$ represents the change in entropy due to both the energy difference between the two states and nonequilibrium driving. We introduce the parameters $\epsilon$ and $\eta$, which control the symmetric and antisymmetric parts of the transition rates, respectively, without affecting each other. The steady-state probability of the system, denoted by $\pi_i$, satisfies $\sum_{j(\neq i)}(W_{ij}\pi_j - W_{ji}\pi_i) = 0$. The thermodynamic and kinetic aspects of transitions are characterized by the currents $J_{ij} = W_{ij}\pi_j - W_{ji}\pi_i$ and the traffic $a_{ij} = W_{ij}\pi_j + W_{ji}\pi_i$, respectively, in the steady state.

When the antisymmetric parameter $F_{ij}$ sums to a nonzero value along at least one closed path in the state space, the system is driven out of equilibrium and dissipates energy constantly in the steady state. This dissipation is characterized by the (mean) EP rate $\dot{\Sigma} = \sum_{i<j} J_{ij} \ln(W_{ij}\pi_j/W_{ji}\pi_i) = \sum_{i<j} J_{ij}F_{ij}$ [26]. We set the Boltzmann constant to unity throughout. The pseudo-EP rate, a measure of the irreversibility of dynamics, has been found useful in deriving thermodynamic uncertainty relations and is defined as $\dot{\Sigma}_{\rm ps} = \sum_{i<j} 2J_{ij}^2/a_{ij}$ [25,28,29]. The log-mean inequality, $2/(x+y) \le (\ln x - \ln y)/(x-y)$ for positive $x$ and $y$, guarantees that $\dot{\Sigma}_{\rm ps} \le \dot{\Sigma}$. While the EP characterizes the irreversible nature of nonequilibrium systems, the dynamical activity, defined as $\dot{A} = \sum_{i<j} a_{ij}$, serves as a complementary role by describing the time-symmetric aspect of dynamics [30].

To address both the average behavior and fluctuations, we introduce two stochastic quantities: the state identifier $\eta_i(t) = \delta_{s(t),i}$, where $s(t)$ is the state of the system at time $t$, and $N_{ij}(t)$, which denotes the accumulated number of jumps from state $j$ to state $i$ up to time $t$. To investigate the relations between response and fluctuations, we focus on general time-accumulated observables with arbitrary weights $g_i$ and $\Lambda_{ij}$, defined as

$$\Theta(\tau) = \int_0^\tau dt \left(\sum_i g_i \eta_i(t) + \sum_{i\neq j} \Lambda_{ij} \dot{N}_{ij}(t)\right), \tag{3}$$

where $\tau$ is the observation time and $\dot{N}_{ij}(t)$ denotes the rate of change of $N_{ij}(t)$. We will refer to the observables as currentlike if $g_i = 0$ and $\Lambda_{ij} = -\Lambda_{ji}$ for all pairs $(i,j)$ and as state dependent if $\Lambda_{ij} = 0$ for all pairs $(i,j)$.

*Fluctuation-response inequalities*—Suppose the system is initially in the steady state for $t < 0$, and a system parameter $\theta$ is slightly changed at $t = 0$. The transition rates and the probability distributions are then altered as $W'_{ij} = W_{ij} + (\partial_\theta W_{ij})\Delta\theta$ and $p_i(t) = \pi_i + q_i(t)\Delta\theta$, respectively, up to linear order in the change $\Delta\theta$. The mean value of the observable $\Theta(\tau)$, measured from $t = 0$, deviates from the unperturbed value by $\Delta\Theta(\tau) = \langle\Theta(\tau)\rangle - \langle\Theta(\tau)\rangle_0$, where $\langle\bullet\rangle$ and $\langle\bullet\rangle_0$ denote the ensemble averages over perturbed and unperturbed dynamics, respectively. We define the dynamic response with respect to the change in $\theta$ as $R_\theta(\tau) = \lim_{\Delta\theta\to 0}\Delta\Theta(\tau)/\Delta\theta$. The Cramér-Rao bound provides a general relation between the response to the perturbation and the variance of the observable as $R_\theta^2(\tau) \le \mathrm{Var}(\Theta(\tau))\mathcal{I}_\theta(\tau)$, where $\mathcal{I}_\theta(\tau) = -\langle\partial_\theta^2 \ln \mathcal{P}[\{s(t)\}_{t=0}^\tau]\rangle_0$ is the Fisher information of the path probability $\mathcal{P}[\{s(t)\}_{t=0}^\tau]$ for the unperturbed dynamics [21,22], and $\mathrm{Var}(\bullet)$ denotes the variance calculated in unperturbed dynamics. It is important to note that the derivative $\partial_\theta$, used only for notational brevity in the Fisher information, does not apply to the initial distribution since the perturbation does not alter the initial condition.

For multiple perturbation parameters $(\theta_1, \ldots, \theta_K)$, the Cramér-Rao bound generalizes to $\sum_{\alpha,\beta} R_{\theta_\alpha}(\tau)[\mathcal{I}^{-1}(\tau)]_{\theta_\alpha\theta_\beta} R_{\theta_\beta}(\tau) \le \mathrm{Var}[\Theta(\tau)]$ with the Fisher information matrix whose elements are given by $\mathcal{I}_{\theta_\alpha\theta_\beta}(\tau) = -\langle\partial_{\theta_\alpha}\partial_{\theta_\beta} \ln \mathcal{P}[\{s(t)\}_{t=0}^\tau]\rangle_0$ [31]. When the set of perturbation parameters consists of either the symmetric parameters $B_{ij}$ or the antisymmetric parameters $F_{ij}$, the Fisher information matrix becomes diagonal, leading to the following inequalities (see End Matter):

$$\sum_{i<j} \frac{R_{B_{ij}}^2(\tau)}{\tau a_{ij}} \le \mathrm{Var}[\Theta(\tau)], \tag{4}$$

$$\sum_{i<j} \frac{4R_{F_{ij}}^2(\tau)}{\tau a_{ij}} \le \mathrm{Var}[\Theta(\tau)], \tag{5}$$

where the summation is taken only over pairs $i < j$ due to the conditions $B_{ij} = B_{ji}$ and $F_{ij} = -F_{ji}$. We will refer to perturbations in $B_{ij}$ and $F_{ij}$ as kinetic and entropic perturbations, respectively. This terminology is motivated by the fact that, in relaxation dynamics to equilibrium, the former changes the timescale of relaxation without affecting the equilibrium distribution, whereas the latter modifies the entropy of the system at the equilibrium. For observables in the set $\mathcal{S}$, which includes currentlike observables, state-dependent observables, and their combinations with the constraint $\Lambda_{ij} = -\Lambda_{ji}$, the dynamic response to the symmetric parameter $B_{ij}$ and that to the antisymmetric parameter $F_{ij}$ are interrelated by the identity

$$\frac{R_{B_{ij}}(\tau)}{R_{F_{ij}}(\tau)} = \frac{2J_{ij}}{a_{ij}} \tag{6}$$

for all observation times $\tau$ (see Supplemental Material [32] for the derivation). Plugging this identity into (5), we obtain another inequality involving the kinetic response,

$$\sum_{i<j} \frac{a_{ij} R^2_{B_{ij}}(\tau)}{\tau J^2_{ij}} \leq \mathrm{Var}[\Theta(\tau)] \quad \text{for } \Theta \in \mathcal{S}. \tag{7}$$

We will refer to the inequalities (4), (5), and (7) as fluctuation-response inequalities (FRIs) following [13], where a general inequality between fluctuations and response is proposed based on an information-theoretic approach.

The FRIs generalize the FRRs discovered in [20] in two ways. First, these inequalities are valid for all observation times $\tau$ and encompass the FRRs as the dynamic-response function $R_\theta(\tau)$ converges to the static-response function in the limit $\tau \to \infty$. Second, unlike the FRRs, which are applicable only to currentlike observables, the FRIs allow general observables for (4) and (5) and observables in the set $\mathcal{S}$ for (7). While the FRRs are equalities derived from extensive linear algebraic steps, the FRIs are inequalities resulting from a straightforward application of the Cramér-Rao bound.

Figure 1 illustrates the validity of the FRIs (4) and (5) in four-state Markov jump processes with various topologies and system parameters. The vertical axes in all figures represent $\mathcal{Q}$, a quantity obtained by transforming inequalities into the form $\mathcal{Q} \leq 1$. Unlike currentlike observables as reported in [20], the FRIs (4) and (5) do not appear to converge to equalities in the limit $\tau \to \infty$ for general observables. The validity of FRI (7) for currentlike and state-dependent observables is examined in Fig. 2. The numerical results suggest that the FRI (7) converges to equality in both limits $\tau \to 0$ and $\tau \to \infty$ for currentlike observables and only in the limit $\tau \to \infty$ for state-dependent observables. While the convergence to equality in the limit $\tau \to 0$ is straightforward to verify [32], demonstrating the convergence in the limit $\tau \to \infty$ requires a more sophisticated analysis, as performed in [20]. The inverted bell shape and increasing patterns shown in Fig. 2 emerge due to the fixed boundary values of $\mathcal{Q}$ at $\tau \to 0$ and $\tau \to \infty$. This becomes clearer when examining representative time-dependent curves with fixed transition rates and observables, as presented in Supplemental Material [32]. This illustrates how the scatter patterns in Fig. 2 arise from a superposition of curves with different parameter values. In contrast, since the equality condition at the short- and long-time limits does not hold for general observables, Fig. 1 exhibits more complex patterns.

The limiting behavior of the FRIs can be understood by separately considering the different time regimes. First, in the short-time limit $\tau \to 0$, only a single jump event can occur within an infinitesimal time interval. For currentlike observables, this implies that the currents associated with different edges remain uncorrelated, and the variance decomposes into independent contributions from each edge, i.e., $\mathrm{Var}[\Theta(\tau)] \approx \tau \sum_{i<j} \Lambda^2_{ij} a_{ij}$. The responses are similarly dominated by the direct effect of perturbing the transition rate on the same edge, yielding $R_{B_{ij}}(\tau) \approx \tau \Lambda_{ij} J_{ij}$

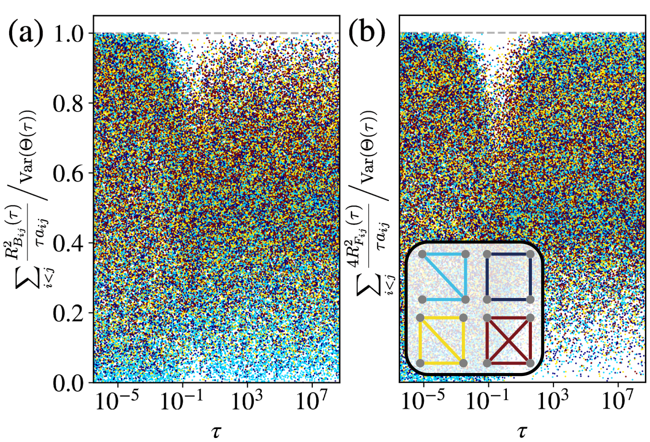


FIG. 1. Numerical verification of FRIs for general observables. (a) and (b) correspond to (4) and (5), respectively. For the symmetric and antisymmetric parameters, $B_{ij}$ and $e^{F_{ij}}$ are randomly sampled from $[-2, 2]$ and $[0, 10]$, respectively. The observation time is given as $\tau = e^x$ where $x$ is drawn randomly from $[-15, 20]$. Weights $g_i$ and $\Lambda_{ij}$ are sampled from $[-2, 2]$. The network topology is randomly selected from the four possible configurations shown in the inset of (b). Different point colors represent results from the respective topologies, matching the colors in the inset. The total number of points is $10^5$.

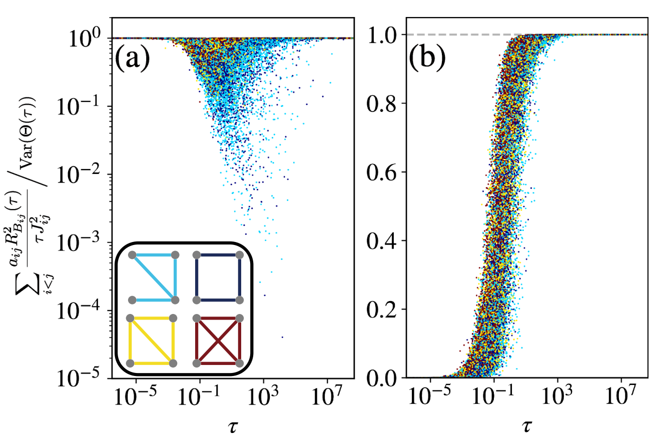


FIG. 2. Numerical verification of FRI (7) for (a) currentlike observables and (b) state-dependent observables. Transition rates, observation times, and weights of observables are sampled within the same ranges as in Fig. 1. The network topology is randomly selected from the four configurations in the inset of (a), with data point colors matching the respective topologies. The total number of points is $10^5$.

and $R_{F_{ij}}(\tau) \approx \frac{1}{2}\tau\Lambda_{ij}a_{ij}$. As a result, both (5) and (7) become equalities in this limit. For state-dependent observables, both the variance and the response functions scale quadratically with time, which explains the vanishing behavior observed in Fig. 2(b) as $\tau \to 0$. Second, in the long-time limit $\tau \to \infty$, where the dynamic responses converge to the static responses, it has been proven that Eqs. (5) and (7) become equalities for both currentlike observables [12] and state-dependent observables [39]. Although a general physical explanation remains elusive, the convergence to the equality in this limit for currentlike observables can be interpreted as a generalization of the fluctuation-dissipation theorem to arbitrary nonequilibrium steady states. This generalized relation reduces to the conventional relationship between Onsager coefficients and equilibrium covariance defined at the edge level near equilibrium [12].

*Response uncertainty relations*—Further applications of the Cauchy-Schwartz inequality to the FRIs lead to the recently discovered R-TUR and its variants [19,20]. The responses of interest are now $R_\epsilon(\tau) = \sum_{i<j} b_{ij}R_{B_{ij}}$ and $R_\eta(\tau) = \sum_{i<j} f_{ij}R_{F_{ij}}$, with the shorthand notations $b_{ij} = d_\epsilon B_{ij}$ and $f_{ij} = d_\eta F_{ij}$. Here, $R_\epsilon$ (respectively, $R_\eta$) denotes the response of an observable with respect to the perturbation of the global parameter $\epsilon(\eta)$, which induces simultaneous variations in all symmetric (antisymmetric) parameters according to their functional dependence on $\epsilon(\eta)$. An explicit example illustrating how global responses arise as combinations of local responses is presented in Supplemental Material [32] using a quantum dot model. Applying the Cauchy-Schwartz inequality $\sum_{i<j}(x_{ij}/y_{ij})^2 \geq (\sum_{i<j} x_{ij})^2/(\sum_{i<j} y_{ij}^2)$ to (4) and (5), we obtain the following results: choosing $x_{ij} = b_{ij}R_{B_{ij}}$ and $y_{ij} = b_{ij}\sqrt{a_{ij}}$ yields

$$\mathrm{Var}[\Theta(\tau)] \geq \frac{R_\epsilon^2(\tau)}{\tau\sum_{i<j} b_{ij}^2 a_{ij}} \geq \frac{R_\epsilon^2(\tau)}{\tau b_{\max}^2 \dot{A}}, \tag{8}$$

while choosing $x_{ij} = f_{ij}R_{F_{ij}}$ and $y_{ij} = f_{ij}\sqrt{a_{ij}}$ yields

$$\mathrm{Var}[\Theta(\tau)] \geq \frac{4R_\eta^2(\tau)}{\tau\sum_{i<j} f_{ij}^2 a_{ij}} \geq \frac{4R_\eta^2(\tau)}{\tau f_{\max}^2 \dot{A}}, \tag{9}$$

where $b_{\max} = \max_{i,j}|b_{ij}|$ and $f_{\max} = \max_{i,j}|f_{ij}|$. These inequalities, called the response kinetic uncertainty relation, show that the ratio of the responses to kinetic or entropic perturbations to fluctuations is bounded from above by the dynamical activity of the unperturbed system.

Similarly, with the choice $x_{ij} = b_{ij}R_{B_{ij}}$ and $y_{ij} = b_{ij}J_{ij}/\sqrt{a_{ij}}$, applying the Cauchy-Schwartz inequality to (7) leads to

$$\mathrm{Var}[\Theta(\tau)] \geq \frac{2R_\epsilon^2(\tau)}{\tau b_{\max}^2 \dot{\Sigma}_{\mathrm{ps}}} \quad \text{for } \Theta \in \mathcal{S}. \tag{10}$$

We refer to this relation as the response thermodynamic-kinetic uncertainty relation (R-TKUR), as it imposes an upper bound on the response that depends on both EP rate (a thermodynamic quantity) and dynamical activity (a kinetic quantity). This can be seen clearly upon noting that the pseudo-EP rate satisfies the following Jensen inequality $\dot{\Sigma}_{\mathrm{ps}} = \sum_{i<j} 2a_{ij}\phi^2[(J_{ij}/2a_{ij})\ln(W_{ij}\pi_j/W_{ji}\pi_i)] \leq 2\dot{A}\phi^2(\dot{\Sigma}/2\dot{A})$ with a concave function $\phi(x) = x/\psi(x)$, where $\psi(x)$ is the inverse function of $x\tanh x$ [25]. One of the two contributions, $\dot{\Sigma}$ or $\dot{A}$, dominates in the limiting cases: for $x \ll 1$, $\phi^2(x) \approx x$ and for $x \gg 1$, $\phi^2(x) \approx 1$. Thus, the R-TKUR reduces to the finite-time version of R-TUR [19],

$$\frac{R_\epsilon^2(\tau)}{\mathrm{Var}[\Theta(\tau)]} \leq \frac{\tau b_{\max}^2 \dot{\Sigma}}{2} \quad \text{for } \Theta \in \mathcal{S}, \tag{11}$$

near equilibrium, where $\dot{\Sigma}/\dot{A} \ll 1$, and reduces to (8) far from equilibrium, where $\dot{\Sigma}/\dot{A} \gg 1$. It is worth noting that (8)–(10) hold for all observation times $\tau$, generalizing the static-response relations derived in the limit $\tau \to \infty$ in [19,20] to dynamic response.

When the kinetic perturbation is applied uniformly, i.e., $b_{ij} = b = b_{\max}\ \forall\ ij$, the response of currentlike observables becomes proportional to the mean value of the observable as $R_\epsilon(\tau) = b\langle\Theta(\tau)\rangle$ [19]. As a result, the R-TKUR in Eq. (10) reproduces the thermodynamic-kinetic uncertainty relation [25,40], which encompasses both the thermodynamic uncertainty relation [41,42] and the kinetic uncertainty relation [43,44].

*Quantum generalization*—The FRI can also be extended to Markovian open quantum systems, whose dynamics are governed by the Lindblad quantum master equation,

$$\dot{\rho}(t) = -i[H, \rho(t)] + \sum_{k=1}^{K} \mathcal{D}[L_k^{\theta_k}]\rho(t), \tag{12}$$

where $\rho(t)$ is the density operator at time $t$, $H$ is the system Hamiltonian, $L_k^{\theta_k}$ are jump operators, $K$ denotes the number of jump operators, and $\mathcal{D}[L]\bullet \equiv L\bullet L^\dagger - \{L^\dagger L, \bullet\}/2$ is a superoperator acting on the operators to its right. The Planck constant $\hbar$ is set to unity throughout. In this section, the notations $[\bullet, \circ]$ and $\{\bullet, \circ\}$ are reserved for the commutator and anticommutator of two quantum operators $\bullet$ and $\circ$, respectively. The jump operators are parametrized as

$$L_k^{\theta_k} = e^{\theta_k/2}L_k, \tag{13}$$

with a set of real-valued parameters $(\theta_1, \theta_2, \ldots, \theta_K)$ that controls the magnitude of each jump operator.

As in the classical case, we consider the system in the steady state characterized by the density operator $\rho_{\mathrm{ss}}$ for $t < 0$, which is then slightly perturbed in its parameters $\theta_k$ at time $t = 0$. The quantum Crameŕ-Rao bound generalizes the

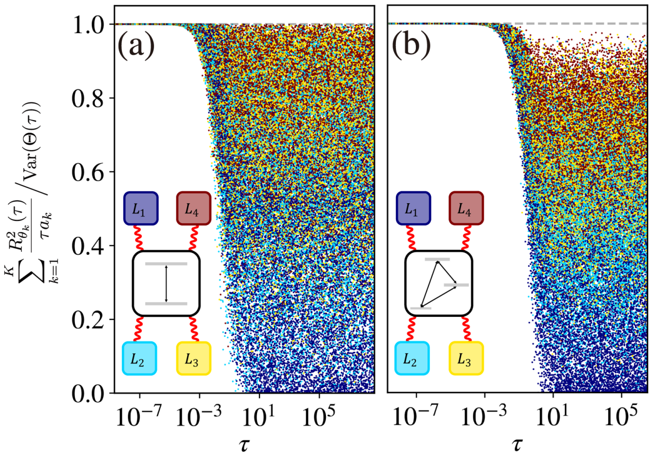


FIG. 3. Numerical verification of Eq. (14) for (a) a two-level system and (b) a three-level system. The Hamiltonian is constructed as $(A + A^\dagger)/2$, where $A$ is a randomly generated matrix with the real and imaginary parts of each element independently and uniformly sampled from $[-1, 1]$. Jump operators are generated similarly without enforcing the Hermitian condition, with the number of jump operators randomly chosen between 1 and 4. Different numbers of jump operators are represented by distinct colors: blue (one jump operator), cyan (two jump operators), yellow (three jump operators), and dark red (four jump operators), respectively. Weights $\Lambda_k$ are randomly sampled from $[-1, 1]$, and the observation time $\tau$ is given by $\tau = e^x$, where $x$ is uniformly sampled from $[-15, 20]$.

classical one as $\sum_{\alpha,\beta} R_{\theta_\alpha}(\tau)[\mathcal{I}_Q^{-1}(\tau)]_{\theta_\alpha\theta_\beta} R_{\theta_\beta}(\tau) \le \mathrm{Var}[\Theta(\tau)]$, where $\mathcal{I}_Q(\tau)$ is the quantum Fisher information, defined as the maximum classical Fisher information over all positive operator-valued measures [45,46]. Similar to the classical case, the quantum Fisher information matrix with respect to changes in $\theta_k$ becomes diagonal (see End Matter), which yields the quantum version of the FRI for general observables,

$$\sum_{k=1}^{K} \frac{R^2_{\theta_k}(\tau)}{\tau a_k} \le \mathrm{Var}[\Theta(\tau)], \qquad (14)$$

where $a_k = \mathrm{tr}[L_k^{\theta_k} \rho (L_k^{\theta_k})^\dagger]$ is the traffic through the $k$ th jump operator.

The validity of the quantum FRI is examined in Fig. 3 for two- and three-level systems, with randomly generated Hamiltonians and jump operators. The local detailed balance condition is not imposed on the jump operators in order to demonstrate the broad validity of the quantum FRI. In our numerical analysis, we focus on the currentlike observables of the form $\Theta(\tau) = \sum_{k=1}^{K} \Lambda_k N_k(\tau)$, where $N_k(\tau)$ represents the total number of jumps via the $k$ th jump operator up to time $\tau$ and $\Lambda_k$ is the corresponding weight. State-dependent observables are not considered in this plot, as no analytical expression for their variance is available. Similar to Fig. 2(a), the FRIs converge to equality in the limit $\tau \to 0$ (see Supplemental Material [32] for the proof). However, unlike in the classical case, the ratio of both sides of Eq. (14) converges to a finite value less than 1 in the limit $\tau \to \infty$ for a fixed set of control parameters, which results in the overall pattern in Fig. 3 differing from that in Fig. 2(a). We also note that the use of more jump operators results in a tighter inequality, although the precise relationship between the tightness and the number of jump operators requires further investigation. From the quantum FRI, a response uncertainty relation for open quantum systems can be derived straightforwardly,

$$\frac{R^2_\epsilon(\tau)}{\mathrm{Var}[\Theta(\tau)]} \le \tau(\Delta\theta_{\max})^2 \dot{A}, \qquad (15)$$

where $\epsilon$ is the parameter that controls the magnitude of each jump operator via $\theta = \theta(\epsilon)$, $\Delta\theta_{\max} = \max_k\{|d_\epsilon \theta_k|\}$, and $\dot{A} = \sum_k a_k$ is the dynamical activity.

*Conclusion*—In this Letter, we derive FRIs that apply to dynamic response to both kinetic and entropic perturbations, extending the previously established FRRs for static response. The validity of the FRIs extends beyond that of existing FRRs, as they apply to a broader class of observables, including both current-like and state-dependent types, measured over finite times. The FRIs are further extended to open quantum systems governed by the Lindblad quantum master equation. The resulting quantum FRI involves only the dynamical activity. It remains an open question whether similar relations can be found that incorporate the EP, the thermodynamic aspect of nonequilibrium systems.

*Note added*—Recently, two related papers have been published. Reference [47] reports a result similar to Eq. (15), and Ref. [48] presents results comparable to Eqs. (8) and (9).

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## End Matter

*Appendix: Derivation of fluctuation-response inequalities*—Here, we derive the classical and quantum FRIs presented in (4), (5), and (14). In classical Markov jump processes, the probability density of observing the trajectory $\Gamma_\tau = \{s(t)\}_{t=0}^{\tau}$ up to time $\tau$ is given by $\mathcal{P}[\Gamma_\tau] = \pi_{s(0)} e^{-\mathcal{A}[\Gamma_\tau]}$ with

$$\mathcal{A}[\Gamma_\tau] = \int_0^\tau dt \sum_{i\neq j} \big(\eta_j(t) W_{ij} - \dot{N}_{ij}(t) \ln W_{ij}\big). \tag{A1}$$

The elements of the Fisher information matrix associated with the perturbation parameters $(\theta_1, \ldots, \theta_K)$ are obtained by directly differentiating the path probability, yielding

$$\begin{aligned}\mathcal{I}_{\theta_\alpha\theta_\beta}(\tau) &= -\langle \partial_{\theta_\alpha}\partial_{\theta_\beta} \ln \mathcal{P}[\Gamma_\tau]\rangle_0 = \langle \partial_{\theta_\alpha}\partial_{\theta_\beta} \mathcal{A}[\Gamma_\tau]\rangle_0 \\ &= \int_0^\tau dt \sum_{i\neq j} (\pi_j \partial_{\theta_\alpha}\partial_{\theta_\beta} W_{ij} - \pi_j W_{ij} \partial_{\theta_\alpha}\partial_{\theta_\beta} \ln W_{ij}) \\ &= \tau \sum_{i\neq j} W_{ij}\pi_j (\partial_{\theta_\alpha} \ln W_{ij})(\partial_{\theta_\beta} \ln W_{ij}).\end{aligned} \tag{A2}$$

Note that the initial distribution does not contribute to the Fisher information matrix since the perturbation considered does not alter the initial condition. When the perturbation parameters are chosen as either the symmetric parameters $B_{ij}$ or the antisymmetric parameters $F_{ij}$, the following relations hold:

$$\begin{aligned}\partial_{B_{i'j'}} \ln W_{ij} &= \delta_{ii'}\delta_{jj'} + \delta_{ij'}\delta_{i'j}, \\ \partial_{F_{i'j'}} \ln W_{ij} &= \frac{1}{2}(\delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{i'j}).\end{aligned} \tag{A3}$$

Substituting Eq. (A3) into Eq. (A2), we find that the Fisher information matrix becomes diagonal, with elements

$$\begin{aligned}\mathcal{I}_{B_{ij}B_{i'j'}}(\tau) &= \tau\delta_{ii'}\delta_{jj'}a_{ij}, \\ \mathcal{I}_{F_{ij}F_{i'j'}}(\tau) &= \frac{1}{4}\tau\delta_{ii'}\delta_{jj'}a_{ij}.\end{aligned} \tag{A4}$$

Plugging the Fisher information matrix into the Cramér-Rao bound, $\sum_{\alpha\beta} R_{\theta_\alpha}(\tau)[\mathcal{I}^{-1}(\tau)]_{\theta_\alpha\theta_\beta} R_{\theta_\beta}$, with $\theta_\alpha$ being either $B_{ij}$ or $F_{ij}$, we arrive at the classical FRIs, (4) and (5). The diagonality of the Fisher information matrix remains unaffected even when a vertex-dependent parameter, as considered in [11,12], is included in the parametrization of the transition rates. This is because the derivatives with respect to $B_{ij}$ and $F_{ij}$ are operationally equivalent to $W_{ij}\partial/\partial W_{ij} + W_{ji}\partial/\partial W_{ji}$ and $(1/2)(W_{ij}\partial/\partial W_{ij} - W_{ji}\partial/\partial W_{ji})$, respectively (see Sec. IV of [11] for more discussion). These operational definitions remain unchanged regardless of whether a vertex-dependent parameter is present.

For open quantum systems described by the Lindblad quantum master equation (12), the continuous measurement framework allows the quantum Fisher information matrix to be expressed in terms of the solution of the generalized Lindblad equation $\dot{\rho}(\tau) = \mathcal{L}_{\theta^1\theta^2}\rho(\tau)$ [49], where the superoperator $\mathcal{L}_{\theta^1\theta^2}$ is given as

$$\begin{aligned}\mathcal{L}_{\theta^1\theta^2}\bullet &= -i[H,\bullet] + \sum_{k=1}^{K} L_k^{\theta_k^1} \bullet (L_k^{\theta_k^2})^\dagger \\ &\quad - \frac{1}{2}\sum_{k=1}^{K} \Big[(L_k^{\theta_k^1})^\dagger L_k^{\theta_k^1} \bullet + \bullet (L_k^{\theta_k^2})^\dagger L_k^{\theta_k^2}\Big].\end{aligned} \tag{A5}$$

The unperturbed dynamics are restored by setting $\theta^1 = \theta^2 = \theta$, whose superoperator is denoted by $\mathcal{L}$ hereafter. The initial condition is $\rho(0) = \rho_{\rm ss}$, where $\rho_{\rm ss}$ is the steady-state density operator of the unperturbed dynamics, satisfying $\mathcal{L}\rho_{\rm ss} = 0$. Denoting the solution of Eq. (A5) as $\rho_{\theta^1\theta^2}$, the elements of the quantum Fisher information matrix are expressed as [49]

$$
\begin{aligned}
[\mathcal{I}_Q(\tau)]_{\theta_\alpha\theta_\beta} &= 4\partial^1_\alpha\partial^2_\beta \ln\{\mathrm{tr}[\rho_{\theta^1\theta^2}(\tau)]\}_{\theta^1=\theta^2=\theta} \\
&= 4\{\partial^1_\alpha\partial^2_\beta \mathrm{tr}[\rho_{\theta^1\theta^2}(\tau)] \\
&\quad - \partial^1_\alpha \mathrm{tr}[\rho_{\theta^1\theta^2}(\tau)]\partial^2_\beta \mathrm{tr}[\rho_{\theta^1\theta^2}(\tau)]\}_{\theta^1=\theta^2=\theta},
\end{aligned} \tag{A6}
$$

where $\theta^1$ and $\theta^2$ are $K$-dimensional vectors, and $\partial^{1(2)}_{\alpha(\beta)}$ denotes the derivative with respect to the $\alpha$th ($\beta$th) component of $\theta^1(\theta^2)$.

Using the parametrization (13), we can demonstrate that

$$
\begin{aligned}
(\partial^1_\alpha\mathcal{L})_{\theta^1=\theta^2=\theta}\bullet &= \frac{1}{2}[L^{\theta_\alpha}_\alpha \bullet (L^{\theta_\alpha}_\alpha)^\dagger - (L^{\theta_\alpha}_\alpha)^\dagger L^{\theta_\alpha}_\alpha \bullet], \\
(\partial^2_\beta\mathcal{L})_{\theta^1=\theta^2=\theta}\bullet &= \frac{1}{2}[L^{\theta_\beta}_\beta \bullet (L^{\theta_\beta}_\beta)^\dagger - \bullet (L^{\theta_\beta}_\beta)^\dagger L^{\theta_\beta}_\beta]
\end{aligned} \tag{A7}
$$

are traceless maps. From Eq. (A7), we can show that the second term in Eq. (A6) vanishes because

$$
\begin{aligned}
&\partial^{1(2)}_{\alpha(\beta)} \mathrm{tr}[\rho_{\theta^1\theta^2}(\tau)]_{\theta^1=\theta^2=\theta} \\
&= \int_0^\tau dt\ \mathrm{tr}\left[e^{\mathcal{L}(\tau-t)}(\partial^{1(2)}_{\alpha(\beta)}\mathcal{L})_{\theta^1=\theta^2=\theta}e^{\mathcal{L}t}\rho_{\mathrm{ss}}\right] \\
&= \int_0^\tau dt\ \mathrm{tr}\left[(\partial^{1(2)}_{\alpha(\beta)}\mathcal{L})_{\theta^1=\theta^2=\theta}\rho_{\mathrm{ss}}\right] = 0,
\end{aligned} \tag{A8}
$$

where we use the fact that $e^{\mathcal{L}(\tau-t)}$ is a trace-preserving map and that $e^{\mathcal{L}t}\rho_{\mathrm{ss}} = \rho_{\mathrm{ss}}$ in the second equality.

The first term in Eq. (A6) is evaluated as

$$
\begin{aligned}
\partial^1_\alpha\partial^2_\beta \mathrm{tr}[\rho_{\theta^1\theta^2}(\tau)]_{\theta^1=\theta^2=\theta} &= \int_0^\tau dt\, \mathrm{tr}\left[e^{\mathcal{L}(\tau-t)}(\partial^1_\alpha\partial^2_\beta\mathcal{L})_{\theta^1=\theta^2=\theta}e^{\mathcal{L}t}\rho_{\mathrm{ss}}\right] + \int_0^\tau dt\, \mathrm{tr}\left[(\partial^1_\alpha e^{\mathcal{L}(\tau-t)})_{\theta^1=\theta^2=\theta}(\partial^2_\beta\mathcal{L})_{\theta^1=\theta^2=\theta}e^{\mathcal{L}t}\rho_{\mathrm{ss}}\right] \\
&\quad + \int_0^\tau dt\, \mathrm{tr}\left[e^{\mathcal{L}(\tau-t)}(\partial^2_\beta\mathcal{L})_{\theta^1=\theta^2=\theta}(\partial^1_\alpha e^{\mathcal{L}t})_{\theta^1=\theta^2=\theta}\rho_{\mathrm{ss}}\right].
\end{aligned} \tag{A9}
$$

The third integral in Eq. (A9) vanishes as $e^{\mathcal{L}(\tau-t)}$ is trace preserving and $(\partial^2_\beta\mathcal{L})_{\theta^1=\theta^2=\theta}$ is a traceless map. Using the relation

$$
\partial^1_\alpha e^{\mathcal{L}(\tau-t)} = \int_0^{\tau-t} dt' e^{\mathcal{L}(\tau-t-t')}(\partial^1_\alpha\mathcal{L})e^{\mathcal{L}t'}, \tag{A10}
$$

we can show that the second integral in Eq. (A9) also vanishes. Thus, Eq. (A6) simplifies to

$$
\partial^1_\alpha\partial^2_\beta \mathrm{tr}[\rho_{\theta^1\theta^2}(\tau)]_{\theta^1=\theta^2=\theta} = \int_0^\tau dt\, \mathrm{tr}[(\partial^1_\alpha\partial^2_\beta\mathcal{L})_{\theta^1=\theta^2=\theta}\rho_{\mathrm{ss}}]. \tag{A11}
$$

Noting that $(\partial^1_\alpha\partial^2_\beta\mathcal{L})_{\theta^1=\theta^2=\theta} = \delta_{\alpha\beta}L^{\theta_\alpha}_\alpha\rho(L^{\theta_\alpha}_\alpha)^\dagger/4$, the quantum Fisher information matrix becomes diagonal, with elements

$$
[\mathcal{I}_Q(\tau)]_{\theta_\alpha\theta_\beta} = \tau\delta_{\alpha\beta}a_\alpha, \tag{A12}
$$

where $a_\alpha = \mathrm{tr}[L^{\theta_\alpha}_\alpha\rho(L^{\theta_\alpha}_\alpha)^\dagger]$. Substituting the quantum Fisher information matrix into the Cramér-Rao bound, $\sum_{\alpha\beta} R_{\theta_\alpha}(\tau)[\mathcal{I}^{-1}_Q(\tau)]_{\theta_\alpha\theta_\beta}R_{\theta_\beta}$, yields the quantum FRI (14).

# Supplemental Material — Fluctuation-response inequalities for kinetic and entropic perturbations

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## DERIVATION OF EQ. (6) OF THE MAIN TEXT

The key step in deriving Eq. (7) of the main text is proving the identity $R_{B_{ij}}(\tau)/R_{F_{ij}}(\tau) = 2J_{ij}/a_{ij}$ for both current-like and state-dependent observables. This identity can be derived using standard linear response theory [1, 2]. We begin by considering a perturbation in the symmetric parameter $B_{mn}$ shared by the transition rates $W_{mn}$ and $W_{nm}$. Suppose we perturb $B_{mn}$ to $B_{mn} + \Delta B$ at time $t = 0$. The transition rates and the probability distribution are then altered as $W'_{ij} = W_{ij}\big(1 + (\delta_{im}\delta_{jn} + \delta_{in}\delta_{jm})\Delta B\big)$ and $p_i(t) = \pi_i + q_i(t)\Delta B$, respectively, up to linear order in $\Delta B$. Substituting these relations into the master equation yields the equation for $q_i(t)$ as follows:

$$\dot{q}_i(t) = \sum_{j(\neq i)} (W_{ij}q_j(t) - W_{ji}q_i(t)) + \delta_{im}J_{in} + \delta_{in}J_{im} \; . \tag{S1}$$

By introducing a stochastic matrix $\mathsf{W}$, where diagonal components are given by $[\mathsf{W}]_{ii} = -\sum_{j(\neq i)} W_{ji}$ and off-diagonal components by $[\mathsf{W}]_{ij} = W_{ij}$, the linear differential equation for $q_i(t)$ can be solved as

$$q_i(t) = \int_0^t dt' \sum_j [e^{\mathsf{W}(t-t')}]_{ij}(\delta_{jm}J_{jn} + \delta_{jn}J_{jm}) \; . \tag{S2}$$

The time integration in Eq. (S2) can be performed using the spectral decomposition of the stochastic matrix $\mathsf{W}$, expressed as $[\mathsf{W}]_{ij} = \sum_\alpha \lambda_\alpha r_i^\alpha l_j^\alpha$, where $\lambda_\alpha$ are the eigenvalues of $\mathsf{W}$ and $l_i^\alpha$ ($r_i^\alpha$) are the corresponding left (right) eigenvectors normalized as $\sum_i l_i^\alpha r_i^\beta = \delta_{\alpha\beta}$. The unique largest eigenvalue is $\lambda_0 = 0$, with corresponding eigenvectors given by $l_i^0 = 1$ and $r_i^0 = \pi_i$. After carrying out the time integration and rearranging the terms, we obtain

$$q_i(t) = \sum_{\alpha(\neq 0)} \left( \frac{e^{\lambda_\alpha t} - 1}{\lambda_\alpha} \right) r_i^\alpha (l_m^\alpha - l_n^\alpha) J_{mn} \; , \tag{S3}$$

where the eigenvalues are indexed in descending order, i.e., $0 = \lambda_0 > \lambda_1 \geq \lambda_2 \geq \cdots$. The dynamic response of state-dependent observables is then given by

$$\begin{aligned} R_{B_{mn}}(\tau) &= \int_0^\tau dt \sum_i g_i q_i(t) \\ &= J_{mn} \sum_i g_i \big(H_{im}(\tau) - H_{in}(\tau)\big) \; , \end{aligned} \tag{S4}$$

where the function $H_{ij}(\tau)$ is defined as

$$H_{ij}(\tau) \equiv \sum_{\alpha(\neq 0)} \left( \frac{e^{\lambda_\alpha \tau} - 1 - \lambda_\alpha \tau}{\lambda_\alpha^2} \right) r_i^\alpha l_j^\alpha \; . \tag{S5}$$

Since $\lim_{\tau \to 0} H_{ij}(\tau) = 0$, the kinetic response of state-dependent observables vanishes as $\tau \to 0$, as shown in Fig. 2(b) of the main text.

When $B_{mn}$ is perturbed to $B_{mn} + \Delta B$ at time $t = 0$, the current flowing between states $i$ and $j$ changes as $J_{ij}(t) = J_{ij} + K_{ij}(t)\Delta B$ up to linear order in $\Delta B$, where $K_{ij}(t) = (\delta_{im}\delta_{jn} + \delta_{jm}\delta_{in})J_{ij} + W_{ij}q_j(t) - W_{ji}q_i(t)$. In the expression for $K_{ij}(t)$, the first term arises from the change in $W_{ij}$, while the second term results from the change in $p_i(t)$. Substituting Eq. (S3) into $K_{ij}(t)$, we obtain

$$\begin{aligned} K_{ij}(t) &= (\delta_{im}\delta_{jn} - \delta_{jm}\delta_{in})J_{mn} \\ &+ \sum_{\alpha(\neq 0)} \left( \frac{e^{\lambda_\alpha t} - 1}{\lambda_\alpha} \right) (W_{ij} r_j^\alpha - W_{ji} r_i^\alpha)(l_m^\alpha - l_n^\alpha) J_{mn} \, . \end{aligned} \tag{S6}$$

The dynamic response of current-like observables to the kinetic perturbation is then given by

$$\begin{aligned} R_{B_{mn}}(\tau) &= \int_0^\tau dt \sum_{i<j} \Lambda_{ij} K_{ij}(t) \\ &= J_{mn} \left( \tau \Lambda_{mn} + \sum_{i \neq j} \Lambda_{ij} W_{ij} (H_{jm}(\tau) - H_{jn}(\tau)) \right) \, , \end{aligned} \tag{S7}$$

Since $\lim_{\tau \to 0} H_{ij}(\tau)/\tau = 0$, the kinetic response of current-like observables $\mathcal{J}(\tau)$ satisfies $\lim_{\tau \to 0} R_{B_{mn}}(\tau)/\tau = J_{mn}\Lambda_{mn}$. By noting that $\lim_{\tau \to 0} \mathrm{Var}(\mathcal{J}(\tau))/\tau = \sum_{i<j} \Lambda_{ij}^2 a_{ij}$, we confirm that the FRI in Eq. (7) of the main text becomes an equality for current-like observables in the limit $\tau \to 0$ as shown in Fig. 2(a) of the main text.

The analysis for a perturbation in the anti-symmetric parameter $F_{mn}$ is similar to that for $B_{mn}$. When $F_{mn}$ is perturbed to $F_{mn} + \Delta F$ at time $t = 0$, the transition rates are altered as $W'_{ij} = W_{ij}[1 + (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\Delta F/2]$ up to linear order in $\Delta F$. The change of the sign in front of $\delta_{in}\delta_{jm}$ and the factor 1/2 replace only $J_{mn}$ with $a_{mn}/2$ in Eq. (S2). This leads to replacing $J_{mn}$ with $a_{mn}/2$ in Eqs. (S4) and (S7), thereby proving the identity $R_{B_{mn}}(\tau)/R_{F_{mn}}(\tau) = 2J_{mn}/a_{mn}$ for current-like observables and state-dependent observables, and thus their linear combinations. Using the same reasoning as for kinetic perturbations, we find that $\lim_{\tau \to 0} R_{F_{mn}}(\tau) = 0$ for state-dependent observables. Consequently, as $\tau \to 0$, the entropic response vanishes for state-dependent observables, while the FRI in Eq. (5) of the main text becomes an equality for current-like observables.

## ORIGIN OF THE PATTERNS APPEARING IN FIGURES 2 AND 3 OF THE MAIN TEXT

In Figs. 2 and 3 of the main text, we randomly sample the transition rates, observable weights, and observation times to generate scatter plots. In this section, we present results that illustrate the behavior of the fluctuation-response inequalities (FRIs) as a function of observation time, with other parameters held fixed. All equation numbers in this section refer to those in the main text.

Figure S1 shows the behavior of the FRIs over observation time. This figure clarifies that the scatter plots in Figs. 2 and 3 of the main text can be understood as superpositions of various curves corresponding to different parameters. Figure S1(a) shows the numerical verification of Eq. (7) for current-like observables. For fixed transition rates $W_{ij}$ and weights $\Lambda_{ij}$ defining the observable, we typically observe that the ratio of the left-hand side (LHS) to the right-hand side (RHS) of Eq. (7) exhibits an inverted bell-shaped curve. This shape emerges naturally when considering the limiting behavior: as $\tau \to 0$, both the LHS and RHS converge to the same value (denoted $A_0$); similarly, as $\tau \to \infty$, they both approach another common value ($A_\infty$). Therefore, both sides remain approximately constant near $A_0$ in the small-$\tau$ regime, deviate in the intermediate regime, and eventually saturate at $A_\infty$ for large $\tau$. A larger difference between $A_0$ and $A_\infty$ results in a more pronounced dip in the intermediate $\tau$ regime.

For state-dependent observables, the trend is simpler. Since the ratio of the LHS to the RHS vanishes as $\tau \to 0$ and converges to 1 as $\tau \to \infty$, one generally expects the ratio to increase with observation time, although some oscillatory behavior can be observed in Fig. S1(b). The upward trend with observation time underlies the scatter plot in Fig. 2(b) of the main text.

Furthermore, Figs. S1 (c) and (d) show that the quantum FRI in Eq. (14) does not converge to an equality in the limit $\tau \to \infty$. We observe a general trend in which the long-time quantum FRI becomes looser with an increasing number of energy states and a decreasing number of jump operators. This trend is also consistent with the results presented in Fig. 3 of the main text.

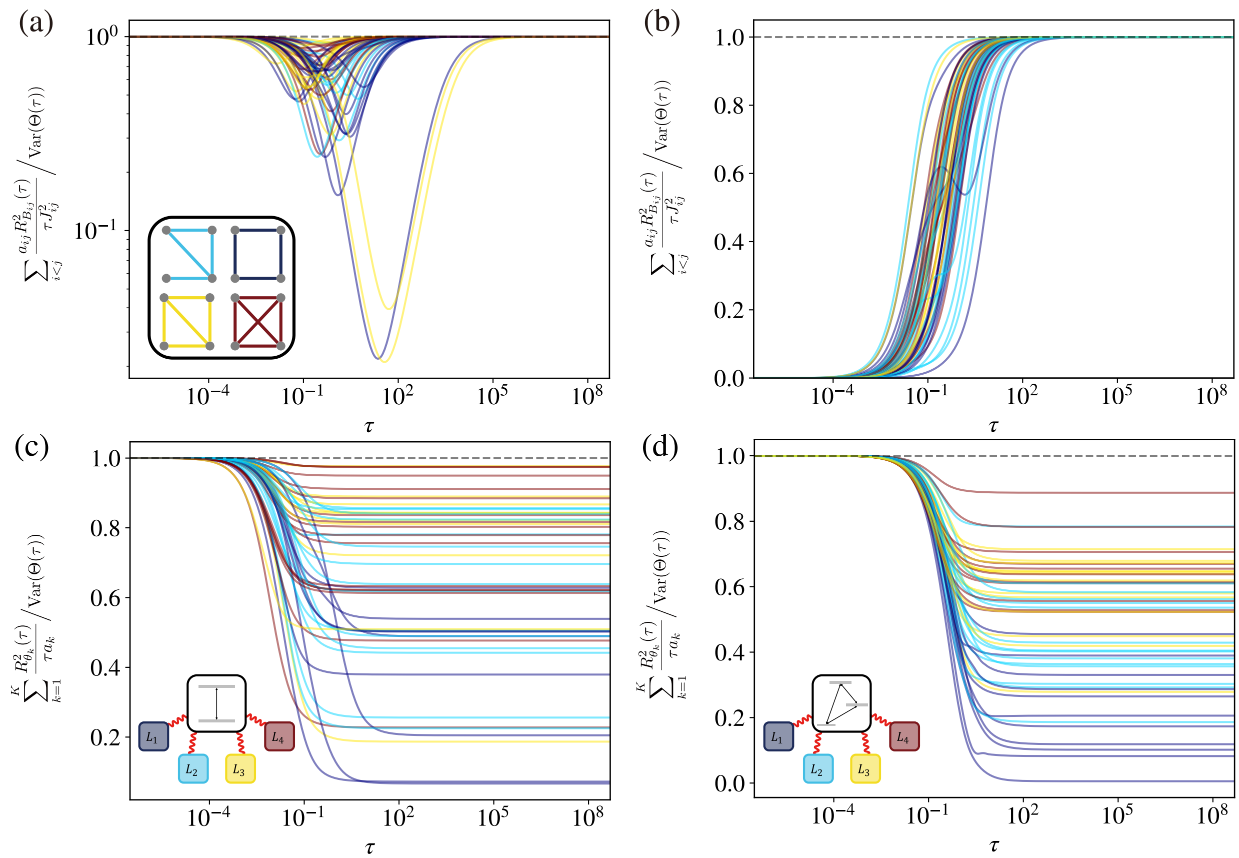


FIG. S1. Numerical verification of Eq. (7) for (a) current-like observables and (b) state-dependent observables, and of Eq. (14) for (c) a 2-level system and (d) a 3-level system. (a,b) The symmetric and anti-symmetric parameters, $B_{ij}$ and $e^{F_{ij}}$, are randomly sampled from $[-2, 2]$ and $[0, 10]$, respectively. The weights $g_i$ and $\Lambda_{ij}$ are sampled from $[-2, 2]$. The network topology is randomly selected from the four possible configurations shown in the inset of panel (a), with the color of each line indicating the corresponding network topology. (c,d) The Hamiltonian is constructed as $(A + A^\dagger)/2$, where $A$ is a random matrix with the real and imaginary parts of each element independently and uniformly sampled from $[-1, 1]$. Jump operators are generated similarly without imposing the Hermitian condition. Different numbers of jump operators are represented by different colors. The weights $\Lambda_k$ are randomly sampled from $[-1, 1]$. Each panel contains 50 lines.

## ILLUSTRATIVE EXAMPLE: TWO-STATE MODEL OF A QUANTUM DOT

In this section, we analyze a minimal example based on the classical description of a quantum dot connected to two reservoirs, aiming to clarify the physical meaning of the FRIs. This is the simplest analytically tractable model that nevertheless provides physical

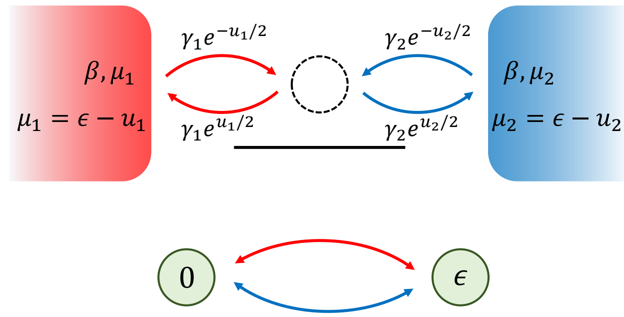


FIG. S2. Schematic of a single-level quantum dot coupled to two reservoirs. The red (blue) arrows indicate electron transfer between the quantum dot and the red (blue) reservoir.

intuition about the nature of perturbations and the saturation behavior of the inequalities.

We consider a single-level quantum dot whose energy is given by $\epsilon$ when it is occupied by an electron and 0 when it is unoccupied. The quantum dot exchanges electrons with two reservoirs at the same temperature $\beta^{-1}$ but with different chemical potentials, $\mu_1 = \epsilon - u_1$ and $\mu_2 = \epsilon - u_2$. Throughout this section, we set the energy unit such that $\beta = 1$. A classical description of this model treats the time-evolution of the occupation probability via the master equation: $\dot{p}_i(t) = \sum_{j=0,1} \sum_{\nu=1,2} (W_{ij}^{\nu} p_j(t) - W_{ji}^{\nu} p_i(t))$ where $p_i(t)$ is the probability that the quantum dot is in state $i$ ($i = 0$: unoccupied, $i = 1$: occupied), and $W_{ij}^{\nu}$ is the transition rate from state $j$ to $i$ due to the exchange of an electron with $\nu$-th reservoir. Imposing local detailed balance, $W_{10}^1 / W_{01}^1 = e^{-u_1}$ and $W_{10}^2 / W_{01}^2 = e^{-u_2}$, we assign the transition rates as $W_{10}^1 = \gamma_1 e^{-u_1/2}$, $W_{01}^1 = \gamma_1 e^{u_1/2}$, $W_{10}^2 = \gamma_2 e^{-u_2/2}$, and $W_{01}^2 = \gamma_2 e^{u_2/2}$, where $\gamma_1$ and $\gamma_2$ are the tunneling rates. The master equation can be written in matrix form as

$$\frac{d}{dt} \begin{pmatrix} p_0(t) \\ p_1(t) \end{pmatrix} = \begin{pmatrix} -\gamma_1 e^{-u_1/2} - \gamma_2 e^{-u_2/2} & \gamma_1 e^{u_1/2} + \gamma_2 e^{u_2/2} \\ \gamma_1 e^{-u_1/2} + \gamma_2 e^{-u_2/2} & -\gamma_1 e^{u_1/2} - \gamma_2 e^{u_2/2} \end{pmatrix} \begin{pmatrix} p_0(t) \\ p_1(t) \end{pmatrix}. \tag{S8}$$

By solving the master equation with respect to the two initial conditions $p_0(0) = 1 - p_1(0) = 0$ and $p_0(0) = 1 - p_1(0) = 1$, we obtain the explicit expressions for the propagator, i.e., the conditional transition probabilities $P(i, t|j, 0)$, as

$$\begin{aligned} P(0,t|0,0) &= \pi_0 + \pi_1 e^{-\phi t}, \quad P(1,t|0,0) = \pi_1 (1 - e^{-\phi t}), \\ P(0,t|1,0) &= \pi_0 (1 - e^{-\phi t}), \quad P(1,t|1,0) = \pi_1 + \pi_0 e^{-\phi t}, \end{aligned} \tag{S9}$$

where $\pi_0 = 1 - \pi_1 = (\gamma_1 e^{u_1/2} + \gamma_2 e^{u_2/2})/\phi$ are the steady-state probability and $\phi = 2\{\gamma_1 \cosh(u_1/2) + \gamma_2 \cosh(u_2/2)\}$.

We first discuss the response of a state-dependent observable to symmetric perturbations on the transition rates. The observable is the accumulated occupation time of the quantum dot, defined as $n(\tau) = \int_0^\tau \eta(t)dt$, where $\eta(t)$ takes the value 1 if the quantum dot is occupied at time $t$, and 0 otherwise. In the steady state, the mean and variance of $n(\tau)$ are given by $\langle n(\tau)\rangle = \pi_1\tau$ and

$$\begin{aligned}\mathrm{Var}(n(\tau)) &= \int_0^\tau dt \int_0^\tau dt'\ \langle \eta(t)\eta(t')\rangle - \langle n(\tau)\rangle^2 \\ &= 2\int_0^\tau dt \int_0^t dt'\ P(1, t-t'|1,0)\pi_1 - (\pi_1\tau)^2 \\ &= \frac{2\pi_0\pi_1(\phi\tau - 1 + e^{-\phi\tau})}{\phi^2}.\end{aligned} \tag{S10}$$

In the second line, we use (i) the time-homogeneity of the dynamics, $P(1,t|1,t') = P(1,t-t'|1,0)$, and (ii) the fact that the contributions from the regions $t > t'$ and $t < t'$ are equal. In this example, the symmetric parameters of the transition rates (denoted $B_{mn}$ in the main text) correspond to $\ln\gamma_1$ and $\ln\gamma_2$. The static response function is given by

$$\frac{\partial\langle n(\tau)\rangle}{\partial \ln\gamma_1} = -\frac{\partial\langle n(\tau)\rangle}{\partial \ln\gamma_2} = \frac{2\gamma_1\gamma_2\tau}{\phi^2}\sinh\left(\frac{u_2 - u_1}{2}\right), \tag{S11}$$

while the dynamic response function, which does not perturb the initial condition, is

$$R_{\ln\gamma_\nu}(\tau) = \int_0^\tau dt \sum_{j=0,1} \frac{\partial P(1,t|j,0)}{\partial \ln\gamma_\nu}\pi_j = \frac{\partial\langle n(\tau)\rangle}{\partial \ln\gamma_\nu}\left(\frac{\phi\tau - 1 + e^{-\phi\tau}}{\phi\tau}\right), \tag{S12}$$

and it converges to the static one as $\tau \to \infty$. The FRI from Eq. (7) of the main text becomes

$$\sum_{\nu=1,2} \frac{a_\nu R^2_{\ln\gamma_\nu}(\tau)}{\tau J_\nu^2} \le \mathrm{Var}(n(\tau)), \tag{S13}$$

where $J_\nu = W^\nu_{10}\pi_0 - W^\nu_{01}\pi_1$ and $a_\nu = W^\nu_{10}\pi_0 + W^\nu_{01}\pi_1$ denote the current and traffic associated with reservoir $\nu$, respectively. The response function is proportional to the current $J_1 = -J_2 = 2\gamma_1\gamma_2 \sinh((u_2 - u_1)/2)/\phi$, so the ratio $R_{\ln\gamma_\nu}(\tau)/J_\nu = (\phi\tau - 1 + e^{-\phi\tau})/\phi^2$ is independent of $\nu$. Moreover, since $a_1 + a_2 = (W^1_{10} + W^2_{10})\pi_0 + (W^1_{01} + W^2_{01})\pi_1 = 2\phi\pi_0\pi_1$, the left-hand side of (S13) simplifies to

$$\sum_{\nu=1,2} \frac{a_\nu R^2_{\ln\gamma_\nu}(\tau)}{\tau J_\nu^2} = \frac{2\pi_0\pi_1(\phi\tau - 1 + e^{-\phi\tau})^2}{\tau\phi^3}. \tag{S14}$$

Comparing this expression with the variance in (S10), the FRI in (S13) reduces to the inequality $\mathcal{Q}_d \equiv \sum_{\nu=1,2} \frac{a_\nu R^2_{\ln\gamma_\nu}(\tau)}{\tau J_\nu^2}/\mathrm{Var}\big(n(\tau)\big) = f(\phi\tau) \le 1$, where $f(x) = (x - 1 + e^{-x})/x$. The function $f(x)$ is monotonically increasing and bounded between 0 and 1 for $x \ge 0$. In the short-time limit ($\tau \to 0$), the left-hand side of the FRI in (S14) vanishes as $\tau^3$, while the variance vanishes as $\tau^2$, so their ratio also vanishes. In contrast, in the long-time limit ($\tau \to \infty$), both sides of the FRI diverge linearly in $\tau$ with the same prefactor, yielding equality. Finally, we note that in this example, the FRI would be violated at all times if the static response function were used. This is because replacing the dynamic response function with the static one makes the ratio $\mathcal{Q}_s \equiv \sum_\nu \frac{a_\nu(\partial\langle n(\tau)\rangle/\partial\ln\gamma_\nu)^2}{\tau J_\nu^2}/\mathrm{Var}\big(n(\tau)\big) = 1/f(\phi\tau)$, which is always larger than or equal to unity for all $\tau \ge 0$.

We next consider the response of a current-like observable to anti-symmetric perturbations on the transition rates. Specifically, we choose the observable as the net number of electrons transferred from reservoir 1 to the quantum dot, denoted as $j(\tau) = \int_0^\tau \big(\dot{N}_{+1}(t) - \dot{N}_{-1}(t)\big)dt$, where $N_{+1}(t)$ [resp. $N_{-1}(t)$] is a piecewise constant function that increases by one whenever an electron is transferred from [resp. to] reservoir 1. In the steady state, the mean and variance of $j(\tau)$ are given by $\langle j(\tau)\rangle = J_1\tau$ and

$$\begin{aligned}
\mathrm{Var}\big(j(\tau)\big) &= \int_0^\tau dt \int_0^\tau dt' \ \langle\{\dot{N}_{+1}(t) - \dot{N}_{-1}(t)\}\{\dot{N}_{+1}(t') - \dot{N}_{-1}(t')\}\rangle - \langle j(\tau)\rangle^2 \\
&= a_1\tau - J_1^2\tau^2 \\
&\quad + 2\int_0^\tau dt \int_0^t dt' \ \big(W^1_{10}P(0,t-t'|1,0)W^1_{10}\pi_0 + W^1_{01}P(1,t-t'|0,0)W^1_{01}\pi_1\big) \\
&\quad - 2\int_0^\tau dt \int_0^t dt' \ \big(W^1_{01}P(1,t-t'|1,0)W^1_{10}\pi_0 + W^1_{01}P(0,t-t'|0,0)W^1_{01}\pi_1\big) \\
&= a_1\tau - \frac{2\{a_1^2 + \gamma_1^2(\pi_0-\pi_1)^2\}(\phi\tau - 1 + e^{-\phi\tau})}{\phi^2},
\end{aligned} \tag{S15}$$

Here, $a_1\tau$ in the second line originated from the contribution of uncorrelated single jump events, and dominates the variance in the short-time limit where at most one electron transfer can occur. In this case, the anti-symmetric parameters of the transition rates (denoted $F_{mn}$ in the main text) correspond to $u_1$ and $u_2$. The static response functions are given by

$$\frac{\partial\langle j(\tau)\rangle}{\partial u_1} = -\frac{a_1\gamma_2\tau}{\phi}\cosh\left(\frac{u_2}{2}\right), \quad \frac{\partial\langle j(\tau)\rangle}{\partial u_2} = \frac{a_2\gamma_1\tau}{\phi}\cosh\left(\frac{u_1}{2}\right), \tag{S16}$$

while the dynamic response functions, which do not perturb the initial condition, are

$$\begin{aligned} R_{u_\nu}(\tau) &= \int_0^\tau dt \sum_{j=0,1} \pi_j \frac{\partial}{\partial u_\nu}\{W_{10}^1 P(0,t|j,0) - W_{01}^1 P(1,t|j,0)\} \\ &= \frac{\partial \langle j(\tau)\rangle}{\partial u_\nu} - a_\nu \gamma_1 \cosh\left(\frac{u_1}{2}\right)\left(\frac{1-e^{-\phi\tau}}{\phi^2}\right) \end{aligned} \tag{S17}$$

and it converges to the static one as $\tau \to \infty$. The FRI from Eq. (5) of the main text becomes

$$\sum_{\nu=1,2} \frac{4R_{u_\nu}^2}{\tau a_\nu} \le \mathrm{Var}\big(j(\tau)\big). \tag{S18}$$

To avoid unwieldy expressions, we consider the simple case of $\gamma_1 = \gamma_2 = \gamma$ and $u_1 = -u_2 = u$ to clearly illustrate the $\tau$-dependence of the FRI. Under these conditions, the response-to-traffic ratio simplifies to $R_{u_\nu}(\tau)/a_\nu = \big((-1)^\nu \phi\tau - 1 + e^{-\phi\tau}\big)/4\phi$. Moreover, since $a_1 = a_2 = \phi/4 = \gamma\cosh(u/2)$ and $\pi_0 = \pi_1 = 1/2$, the left-hand side of (S18) becomes

$$\sum_{\nu=1,2} \frac{4R_{u_\nu}^2}{\tau a_\nu} = \frac{\phi^2\tau^2 + (1-e^{-\phi\tau})^2}{8\phi\tau} \tag{S19}$$

and the right-hand side becomes

$$\mathrm{Var}\big(j(\tau)\big) = \frac{\phi\tau + 1 - e^{-\phi\tau}}{8}. \tag{S20}$$

Comparing both sides, the FRI in (S18) reduces to the inequality $g(\phi\tau) \le 1$, where $g(x) = \{x^2 + (1-e^{-x})^2\}/\{x(x+1-e^{-x})\}$. This function indeed satisfies $g(x) \le 1$ for all $x \ge 0$. Note that both sides of the FRI in (S18) exhibit the same leading-order behavior in both the short-time and long-time limits, resulting in equalities.

As a side note, we remark that the global response functions $R_\epsilon$ and $R_\eta$ defined in the main text are not independent physical quantities, but simply linear combinations of local response functions with respect to all symmetric or anti-symmetric parameters. In the present model, a symmetric global perturbation corresponds to perturbing tunneling rates $\gamma_1$ and $\gamma_2$ simultaneously, each as a function of a single parameter $\epsilon$. For instance, suppose there exists a single parameter such that $\gamma_1 \propto \epsilon$ and $\gamma_2 \propto e^\epsilon$, then a small shift $\epsilon \to \epsilon + \Delta\epsilon$ results in $\ln\gamma_1 \mapsto \ln\gamma_1 + \Delta\epsilon/\epsilon$ and $\ln\gamma_2 \mapsto \ln\gamma_2 + \Delta\epsilon$. Consequently, the global response to a perturbation of $\epsilon$ is given by $R_\epsilon(\tau) = R_{\ln\gamma_1}(\tau)/\epsilon + R_{\ln\gamma_2}(\tau)$. This illustrates how the global response can be interpreted as a weighted sum of local responses, where the weights are determined by how the symmetric parameters or anti-symmetric parameters vary with respect to $\epsilon$ or $\eta$.

## PROOF OF THE EQUALITY OF THE QUANTUM FLUCTUATION-RESPONSE INEQUALITY IN THE SHORT OBSERVATION-TIME LIMIT

In this section, we show that the quantum fluctuation-response inequality (Eq. (14) in the main text) converges to an equality in the short observation-time limit. We consider an open quantum system governed by the Lindblad quantum master equation

$$\dot{\rho}(t) = -i[H, \rho(t)] + \sum_{k=1}^{K} \mathcal{D}[L_k^{\theta_k}]\rho(t) \equiv \mathcal{L}_\theta \rho(t) \ , \tag{S21}$$

where $L_k^{\theta_k}$ are jump operators parametrized by $L_k^{\theta_k} = e^{\theta_k/2} L_k$ and $\mathcal{D}[L]\bullet = L \bullet L^\dagger - \{L^\dagger L, \bullet\}/2$ is a superoperator acting on the operator to its right. To calculate the variance of the observable, we introduce the concept of unraveling, in which the density matrix is conditioned on the stochastic measurement outcomes [3–6]. Through unraveling, we obtain the stochastic evolution of the density operator

$$\rho_c(t+dt) = \rho_c(t) + \mathcal{L}_\theta \rho_c(t) dt + \sum_{k=1}^{K} [dN_k(t) - \mathrm{tr}(\mathcal{M}_k^{\theta_k} \rho_c(t)) dt] \left( \frac{\mathcal{M}_k^{\theta_k} \rho_c(t)}{\mathrm{tr}(\mathcal{M}_k^{\theta_k} \rho_c(t))} - \rho_c(t) \right) \ , \tag{S22}$$

which is called the Belavkin equation [6, 7]. Here, $\mathcal{M}_k^{\theta_k} \bullet \equiv L_k^{\theta_k} \bullet (L_k^{\theta_k})^\dagger$ and $dN_k(t)$ is a classical random variable that takes the value 1 when a jump occurs by the $k$-th jump operator, and 0 otherwise. The unconditional density operator $\rho(t)$ is recovered by averaging over the ensemble of all possible sequences of trajectories. When conditioned on $\rho_c(t)$, the probability of a jump occurring at time $t$ is given by

$$P\big(dN_k(t) = 1 \mid \rho_c(t)\big) = \mathrm{tr}\big(\mathcal{M}_k^{\theta_k} \rho_c(t)\big) dt \ , \tag{S23}$$

while the two-point correlation is given by

$$\left\langle \dot{N}_k(t) \dot{N}_{k'}(t') \right\rangle = \delta_{kk'} \delta(t - t') \mathrm{tr}\big(\mathcal{M}_k^{\theta_k} \rho(t)\big) + \mathrm{tr}\left( \mathcal{M}_k^{\theta_k} e^{\mathcal{L}_\theta (t - t')} \mathcal{M}_{k'}^{\theta_{k'}} \rho(t') \right) \tag{S24}$$

for $t \geq t'$ and

$$\left\langle \dot{N}_k(t) \dot{N}_{k'}(t') \right\rangle = \mathrm{tr}\left( \mathcal{M}_{k'}^{\theta_{k'}} e^{\mathcal{L}_\theta (t' - t)} \mathcal{M}_k^{\theta_k} \rho(t) \right) \tag{S25}$$

for $t < t'$.

We consider current-like observables defined as

$$\Theta(\tau) = \int_0^\tau dt \sum_{k=1}^{K} \Lambda_k \dot{N}_k(t) \tag{S26}$$

with arbitrary weights $\Lambda_k$ and observation time $\tau$. To obtain a compact expression for the steady-state variance

$$\mathrm{Var}(\Theta(\tau)) = \int_0^\tau dt \int_0^\tau dt' \sum_{k,k'=1}^{K} \Lambda_k \Lambda_{k'} \left[\left\langle \dot{N}_k(t)\dot{N}_{k'}(t')\right\rangle - \mathrm{tr}(\mathcal{M}_k^{\theta_k}\rho_{\mathrm{ss}})\mathrm{tr}(\mathcal{M}_{k'}^{\theta_{k'}}\rho_{\mathrm{ss}})\right] , \tag{S27}$$

we first introduce the vectorized notation of density operators and superoperators. In this notation, density operators become $d^2$-dimensional vectors and superoperators become $d^2 \times d^2$ matrices, where $d$ is the dimension of the Hilbert space. With this vectorized notation, the spectral decomposition of the superoperator $\mathcal{L}_\theta$ reads

$$[\mathcal{L}_\theta]_{ij} = \sum_\alpha \lambda_\alpha r_i^\alpha l_j^\alpha, \tag{S28}$$

where $\lambda_\alpha$ are the eigenvalues of $\mathcal{L}_\theta$ and $l_i^\alpha$ ($r_i^\alpha$) are the corresponding left (right) eigenvectors normalized as $\sum_i l_i^\alpha r_i^\beta = \delta_{\alpha\beta}$. The unique largest eigenvalue is $\lambda_0 = 0$, with the corresponding right eigenvector given by the vectorized steady-state density operator $\rho_{\mathrm{ss}}$. The left eigenvector $l_i^0$ takes the value 1 if the index $i$ corresponds to a diagonal element of $\rho_{\mathrm{ss}}$, and 0 otherwise, ensuring that $\sum_i l_i^0 r_i^0 = \mathrm{tr}(\rho_{\mathrm{ss}}) = 1$. We define a superoperator $\mathcal{H}$ from the identity

$$\begin{aligned}\int_0^\tau dt \int_0^t dt' \left[e^{\mathcal{L}_\theta(t-t')}\right]_{ij} &= \int_0^\tau \int_0^t dt' \sum_\alpha e^{\lambda_\alpha(t-t')} r_i^\alpha l_j^\alpha \\ &= \frac{\tau^2}{2}[\rho_{\mathrm{ss}}]_i + \sum_{\alpha:\lambda_\alpha\neq 0} \frac{e^{\lambda_\alpha\tau} - 1 - \lambda_\alpha\tau}{\lambda_\alpha^2} r_i^\alpha l_j^\alpha \\ &\equiv \frac{\tau^2}{2}[\rho_{\mathrm{ss}}]_i + \tau[\mathcal{H}]_{ij},\end{aligned} \tag{S29}$$

where the last equality defines $\mathcal{H}$. The double time integration of $\langle \dot{N}_k(t)\dot{N}_{k'}(t')\rangle$ can be concisely expressed as

$$\begin{aligned}\int_0^\tau dt \int_0^\tau dt' \ \left\langle \dot{N}_k(t)\dot{N}_{k'}(t')\right\rangle = \tau\delta_{kk'}\mathrm{tr}(\mathcal{M}_k^{\theta_k}\rho_{\mathrm{ss}}) + \tau^2 \mathrm{tr}(\mathcal{M}_k^{\theta_k}\rho_{\mathrm{ss}})\mathrm{tr}(\mathcal{M}_{k'}^{\theta_{k'}}\rho_{\mathrm{ss}}) \\ + \tau\left[\mathrm{tr}\left(\mathcal{M}_k^{\theta_k}\mathcal{H}\mathcal{M}_{k'}^{\theta_{k'}}\rho_{\mathrm{ss}}\right) + \mathrm{tr}\left(\mathcal{M}_{k'}^{\theta_{k'}}\mathcal{H}\mathcal{M}_k^{\theta_k}\rho_{\mathrm{ss}}\right)\right]\end{aligned} \tag{S30}$$

Plugging (S30) into (S27), we have

$$\mathrm{Var}(\Theta(\tau)) = \tau\left[\sum_{k=1}^{K} \Lambda_k^2 \mathrm{tr}(\mathcal{M}_k^{\theta_k}\rho_{\mathrm{ss}}) + \sum_{k,k'=1}^{K} \Lambda_k\Lambda_{k'}\left[\mathrm{tr}\left(\mathcal{M}_k^{\theta_k}\mathcal{H}\mathcal{M}_{k'}^{\theta_{k'}}\rho_{\mathrm{ss}}\right) + \mathrm{tr}\left(\mathcal{M}_{k'}^{\theta_{k'}}\mathcal{H}\mathcal{M}_k^{\theta_k}\rho_{\mathrm{ss}}\right)\right]\right] . \tag{S31}$$

Next, we calculate the response of

$$\langle\Theta(\tau)\rangle = \int_0^\tau dt \sum_{k'=1}^{K} \Lambda_{k'} \mathrm{tr}(\mathcal{M}_{k'}^{\theta_{k'}} e^{\mathcal{L}_\theta t}\rho(0)) \ . \tag{S32}$$

to a perturbation in $\theta_k$. We consider the dynamical response of a system initially in the steady state, and thus the perturbation does not alter the initial state. With this in mind, and using $\partial_{\theta_k}\mathcal{M}_{k'}^{\theta_{k'}} = \delta_{kk'}\mathcal{M}_k^{\theta_k}$, we obtain

$$R_{\theta_k}(\tau) = \int_0^\tau dt\ \Lambda_k \mathrm{tr}(\mathcal{M}_k^{\theta_k}\rho_{\mathrm{ss}}) + \sum_{k'=1}^{K} \Lambda_{k'} \mathrm{tr}\left(\mathcal{M}_{k'}^{\theta_{k'}} \left[\int_0^\tau dt\ \partial_{\theta_k} e^{\mathcal{L}_\theta t}\right]\rho_{\mathrm{ss}}\right) \ . \tag{S33}$$

The time integration in the second term can be expressed via the Dyson series expansion

$$\int_0^\tau dt\ \partial_{\theta_k} e^{\mathcal{L}_\theta t} = \int_0^\tau dt \int_0^t dt'\ e^{\mathcal{L}_\theta (t-t')}[\partial_{\theta_k}\mathcal{L}_\theta] e^{\mathcal{L}_\theta t'} \ . \tag{S34}$$

Using the fact that $\rho_{\mathrm{ss}}$ is the steady state, i.e., $e^{\mathcal{L}t'}\rho_{\mathrm{ss}} = \rho_{\mathrm{ss}}$, and that $\partial_{\theta_k}\mathcal{L} = \mathcal{D}[L_k^{\theta_k}]$, we obtain

$$R_{\theta_k}(\tau) = \tau\left[\Lambda_k \mathrm{tr}(\mathcal{M}_k^{\theta_k}\rho_{\mathrm{ss}}) + \sum_{k'=1}^{K} \Lambda_{k'} \mathrm{tr}(\mathcal{M}_{k'}^{\theta_{k'}} \mathcal{H}\mathcal{D}[L_k^{\theta_k}]\rho_{\mathrm{ss}})\right] \ . \tag{S35}$$

Since the eigenvalues except for the unique largest one, $\lambda_0 = 0$, are all negative, the superoperator $\mathcal{H}$ vanishes in the short observation-time limit $\tau \to 0$. Noting that the traffic through the $k$-th channel is defined as $a_k = \mathrm{tr}(\mathcal{M}_k^{\theta_k}\rho_{\mathrm{ss}}) = \mathrm{tr}\big(L_k^{\theta_k}\rho_{\mathrm{ss}}(L_k^{\theta_k})\big)$, and comparing (S31) and (S35), we can prove the quantum fluctuation-response inequality (Eq. (14) in the main text) converges to an equality for current-like observables in the short observation-time limit.


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