Reentrant phase diagram of branching annihilating random walks with one and two offspring

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We investigate the phase diagram of branching annihilating random walks with one and two offspring in one dimension. A walker can hop to a nearest neighbor site or branch with one or two offspring with relative ratio. Two walkers annihilate immediately when they meet. In general, this model exhibits a continuous phase transition from an active state into the absorbing state (vacuum) at a finite hopping probability. We map out the phase diagram by Monte Carlo simulations that shows a reentrant phase transition from vacuum to an active state and finally into vacuum again as the relative rate of the two-offspring branching process increases. This reentrant property apparently contradicts the conventional wisdom that increasing the number of offspring will tend to make the system more active. We show that the reentrant property is due to the static reflection symmetry of two-offspring branching processes and the conventional wisdom is recovered when the dynamic reflection symmetry is introduced instead of the static one.

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I. INTRODUCTION

In recent years, various kinds of nonequilibrium lattice models exhibiting a continuous phase transition from a reactive phase into an absorbing (inactive) phase have been studied extensively [1,2]. Most of the models investigated are found to belong to the directed percolation (DP) universality class [3–6]. A common feature of these models is that the absorbing phase consists of a *single* absorbing state. Even some models with infinitely many absorbing states also exhibit critical behavior in the DP universality class [7,8].

Only a few models have been studied that are not in the DP universality class. They are models A and B of probabilistic cellular automata [9,10], nonequilibrium kinetic Ising models with two different dynamics [11,12], and interacting monomer-dimer models [13–15]. Numerical investigations show that they belong to the same but non-DP universality class. These models share a common property that the absorbing phase consists of two *equivalent* absorbing states.

Recently, the branching annihilating random walks (BAW) have been studied intensively [16–21]. The BAW model is a lattice model where a walker can hop to a nearest-neighbor site with probability p and branch with n offspring in the nearest neighborhood with probability $1-p$. Two walkers annihilate immediately when they meet. In general, this model exhibits a continuous phase transition from an active state into the absorbing state (vacuum) at finite hopping probability p_c. Even though the BAW model has a single absorbing state, its critical property depends on the parity of the number of offspring created in a branching process. Dynamics of the BAW models with even n conserve the number of walkers modulo 2, while the BAW models with odd n evolve without any conservation. The BAW models with odd n exhibit the DP critical behavior [18,19], while the BAW models with even n exhibit the same non-DP behavior as in the models with two equivalent absorbing states [10,13–15,21]. One can find that the total number of kinks are conserved modulo 2 in models with two equivalent absorbing states. Therefore this conservation law may be responsible for the non-DP critical behavior.

The critical exponents characterizing the non-DP behavior are measured accurately by extensive Monte Carlo simulations for the BAW model with $n=4$ in one dimension [21]. Unfortunately, the BAW model with $n=2$, which is simpler, does not have an active phase ($p_c=0$) [17,22]. Recently, ben-Avraham, Leyvraz, and Redner [23] introduced another parameter that controls the two-walker annihilation process and shows that the BAW model with $n=2$ exhibits a continuous phase transition at a finite value of p_c except for the case of infinite annihilation rate (the ordinary BAW model). But the values of the critical exponents are not reported there. As the annihilation rate becomes smaller, it is clear that the system gets more active (more walkers survive) so the critical probability p_c increases.

In the ordinary BAW models, there are two competing elementary processes, i.e., hopping and branching. The hopping process does not increase the number of walkers but can decrease it by two-walker annihilations. As the hopping probability becomes larger, the system tends to trap into the absorbing state (vacuum). The branching process may increase the number of walkers so it can make the system more active. That is why the BAW model with one offspring exhibits a phase transition from an active state into vacuum as p becomes larger. Therefore one may argue that increasing the number of offspring in the branching process will make the system more active so the critical probability p_c will increase as n increases. However, numerical and analytical study of the BAW models with n offspring shows that p_c is not a monotonically increasing function of n. In fact, the values of p_c are 0.1070(5), 0, 0.459(1), 0.7215(5), and 0.718(1) for $n = 1 - 5$ [18,19,21]. These results contradict the conventional wisdom mentioned above and there is no consistent manner in making the system more active by changing the number of offspring.

In order to understand this rather surprising result, we introduce a model that interpolates the BAW models with one offspring and with two offspring. In this model, the branching process creates one offspring or two offspring with relative ratio. We map out the phase diagram by dynamic Monte Carlo simulations that show a reentrant phase transition from vacuum to an active state and finally into vacuum again as the relative rate of the two-offspring process increases at fixed hopping probability. The second phase transition occurs at quite high rates of the two-offspring process (more than 80%). We argue that the second transition is due to the static reflection symmetry of the two-offspring branching process (one offspring to the left and the other to the right of the branching walker: *static branching*). This reentrant second transition disappears and our conventional wisdom is recovered when the dynamic reflection symmetry is introduced (both offspring to the left or to the right of the branching walker with equal probability: *dynamic branching*).

In the next section, we describe the BAW model with one and two offspring with relative ratio. Dynamic Monte Carlo results are discussed that show the reentrant phase diagram. In Sec. III, the BAW model with dynamic branching is introduced. Our numerical results for $n = 2$ find the existence of the non-DP critical behavior at finite hopping probability. In Sec. IV, we study the BAW model with one and two offspring created by dynamic branching. The reentrant behavior disappears entirely as expected. We conclude in Sec. V with a brief summary.

II. THE BAW MODEL WITH ONE AND TWO OFFSPRING: STATIC BRANCHING

We consider the BAW model with one and two offspring with relative ratio in one dimension. The evolution rules of this model are given as follows. First, choose a walker at random. It may hop to a randomly chosen nearest-neighbor site with probability p. If this site is already occupied by another walker, both walkers annihilate immediately. With probability $1 - p$, the randomly chosen walker creates two offspring symmetrically at nearest-neighbor sites with relative probability r ($0 \leq r \leq 1$) or creates one offspring at a randomly chosen nearest-neighbor site with relative probability $1 - r$. When an offspring is created on a site already occupied, both walkers annihilate immediately. The case $r = 0$ corresponds to the BAW model with one offspring, which exhibits a continuous phase transition from an active phase into vacuum at $p \simeq 0.1070$ [19]. The other limiting case $r = 1$ corresponds to the BAW model with two offspring, which does not have an active state at finite values of p [17,22].

We perform dynamic Monte Carlo simulations for this model with various values of $r = 0$, 0.25, 0.50, 0.75, 0.8, 0.85, 0.9, 0.95, and 1. We start with two nearest-neighbor walkers at the central sites of a lattice. Then the system evolves along the dynamical rules of the model. After one attempt of hopping or branching on the average per lattice site (one Monte Carlo step), the time is incremented by one unit. A numer of independent runs, typically 10^5, is made up to 2000 time steps for various values of p near the critical probability p_c. Most runs, however, stop earlier because the system gets into the absorbing state. We measure the survival probability $P(t)$ (the probability that the system is still active at time t), the number of walkers $N(t)$ averaged over all runs, and the mean-square distance of spreading $R^2(t)$ averaged over the surviving runs. At criticality, the values of these quantities scale algebraically in the long-time limit [24]

$$\begin{aligned} P(t) &\sim t^{-\delta}, \\ N(t) &\sim t^{\eta}, \\ R^2(t) &\sim t^{z}, \end{aligned} \tag{1}$$

and double-logarithmic plots of these values against time show straight lines. Off criticality, these plots show some curvature. More precise estimates for the scaling exponents can be obtained by examining the local slopes of the curves. The effective exponent $\delta(t)$ is defined as

$$-\delta(t) = \frac{\log\left[P(t)/P(t/b)\right]}{\log b} \tag{2}$$

and similarly for $\eta(t)$ and $z(t)$. In Fig. 1, we plot the effective exponents against $1/t$ with $b = 5$ for $r = 0.25$. Off criticality these plots show upward or downward curvatures. From Fig. 1, we estimate $p_c = 0.1265(5)$ and dynamic exponents $\delta = 0.158(2)$, $\eta = 0.310(3)$, $z = 1.27(1)$. These values are in an excellent accord with those of the DP universality class; $\delta = 0.1596(4), \eta = 0.3137(10)$, and $z = 1.2660(14)$ (see Ref. [21]). This result supports the conjecture that models with a single absorbing state and no conservation laws should belong to the DP universality class.

Similarly we determine the values of the critical probability p_c and the dynamic exponents for various values of r. As expected from the above conjecture, the values of the dynamic exponents stay almost unchanged except for $r = 1$. Estimates of p_c for various values of r are listed in Table I. The value of p_c slowly increases as r varies up to ~ 0.75 and drastically decreases to zero as r approaches the value of unity. The r-p phase diagram is drawn in Fig. 2 where a reentrant phase transition is explicitly shown. At fixed values of $p = 0.11$–0.14, the system undergoes phase transitions from vacuum to an active state and finally into vacuum again as r becomes larger (more offspring are created). The second transition occurs at quite high rates of the two-offspring branching process. This result implies that our conventional wisdom does not apply near $r = 1$.

The BAW model with two offspring can be solved exactly at $p = 0$, where pairs of nearest-neighbor walkers diffuse like the ordinary random walks, i.e., $\cdots 001100 \cdots \to \cdots 000110 \cdots$ where "1" represents an occupied site and "0" a vacant site. When two pairs collide with each other, they just bounce back. So the number of walkers is bounded in this model, in contrast to the BAW model with one offspring where the number of walkers is not bounded. So our conventional wisdom that the BAW model with more offspring may be more active does not work. For $p > 0$, these pairs annihilate by hopping processes so the system becomes inactive in the long-time limit.

We note that the picture of diffusing pairs is mainly due to the static reflection symmetry of the branching process. If the dynamic reflection symmetry is adopted instead of the static one, this diffusing-pair picture is no longer valid. In this case, a walker branches two offspring both to the left or to the right of itself with equal probability. The reflection symmetry is not broken on average, but this branching process allows a formation of long chains of walkers, i.e., $\cdots 00011000 \cdots \to \cdots 01111000 \cdots$. So the number of walkers is not bounded from above and an active phase may appear at finite hopping probability.

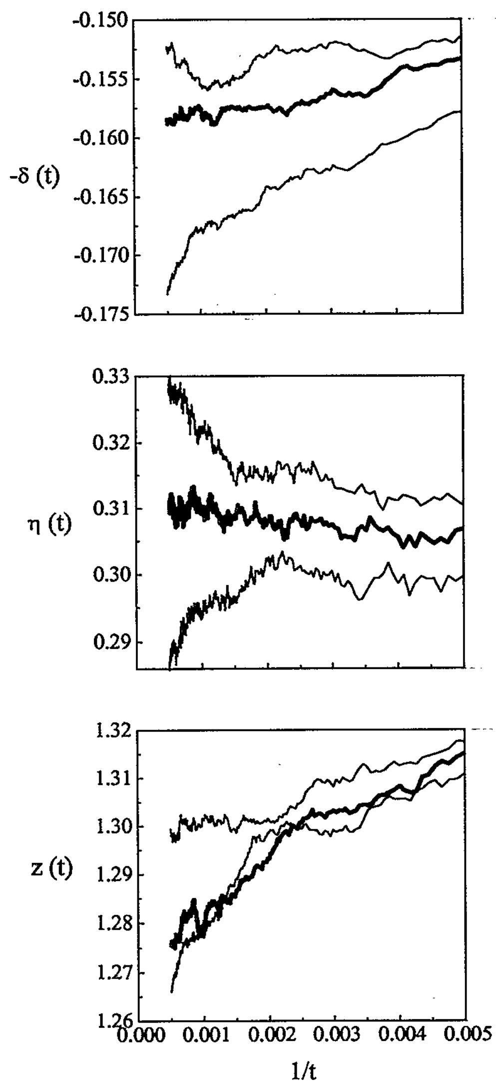

FIG. 1. Plots of the effective exponents against $1/t$. Three curves from top to bottom in each panel correspond to $p = 0.1260$, 0.1265, and 0.1270. Thick lines are critical lines ($p = 0.1265$).

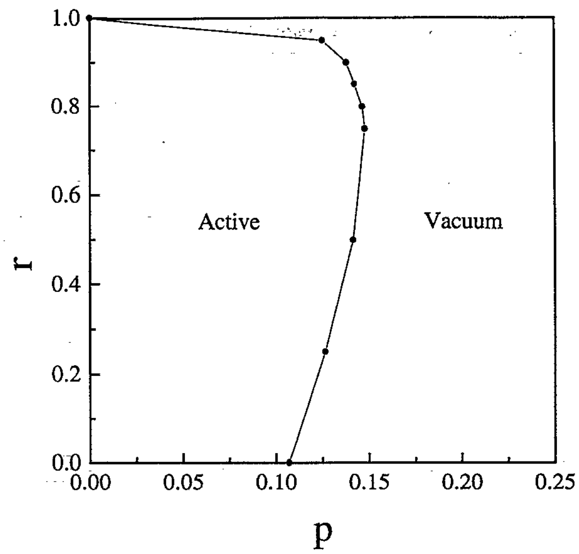

FIG. 2. The r-p phase diagram for the BAW model with one and two offspring with relative ratio (static branching). The full line between the Monte Carlo data is a guide to the eye.

III. THE BAW MODEL WITH TWO OFFSPRING: DYNAMIC BRANCHING

We study the BAW model with two offspring created by dynamic branching. The hopping process is the same as in the ordinary BAW models, but in the branching process a randomly chosen walker creates two offspring both on the sites to the left or to the right of the walker with equal probability.

We perform dynamic Monte Carlo simulations, starting from a pair of walkers at the central sites of a lattice. 10^6 independent runs are made during 2000 time steps and we measure $P(t)$, $N(t)$, and $R^2(t)$. In Fig. 3, we plot the effective exponents $\delta(t)$, $\eta(t)$, and $z(t)$ against $1/t$ with $b = 5$. These plots clearly show the existence of an active phase. We estimate $p_c = 0.5105(7)$ and the dynamic exponents $\delta = 0.287(1)$, $\eta = 0.000(3)$, and $z = 1.155(5)$. These values are in an excellent accord with those of the non-DP universality class; $\delta = 0.285(2)$, $\eta = 0.000(1)$, and $z = 1.141(2)$ for the BAW model with four offspring. This result supports the conjecture that models with particle number conservation of modulo 2 should belong to the same but non-DP universality class.

We also perform dynamic simulations with a different initial configuration. We start with a single walker at the center of a lattice. Conservation of the number of walkers of modulo 2 prevents the system from entering

TABLE I. Critical hopping probability p_c for the BAW model with one and two offspring created by the static branching. r is the relative probability of the two-offspring process. Numbers in parentheses represent the errors in the last digits.

r	0	0.25	0.50	0.75	0.80	0.85	0.90	0.95	1
p_c	0.1070(5)	0.1265(5)	0.1415(5)	0.1480(5)	0.1470(5)	0.1430(5)	0.1380(5)	0.1250(5)	0

the absorbing state (vacuum). So the survival probability exponent δ must be zero. Our numerical results conclude that the dynamic exponents $\eta = 0.283(5)$ and $z = 1.15(1)$. These values also agree very well with those of the BAW model with four offspring [$\eta = 0.282(4)$, $z = 1.140(5)$].

Compared with the ordinary BAW model with two offspring, it is clear that the static reflection symmetry is responsible for the nonexistence of an active phase and the dynamic reflection symmetry makes the system more active. To support this idea, we employ the mean-field theory on spreading of the active region. The difference between these two models lies in the branching process. So we only consider the effect of different branching mechanisms on the boundary of the active region. The boundary can move by branching of walkers at the boundary or nearby. For example, a branching process of a walker at the boundary increases the active region by one unit in the BAW model with the static symmetry. For the BAW model with the dynamic symmetry, the same process increases the active region by the same amount (one unit on average). However, a branching process of a walker near the boundary inside of the active region decreases the active region differently for these two models. Considering all possible cases where the boundary can move and using the mean-field theory to assign a proper probability to each case, we find the outward velocity of the boundary as

$$v_s = (1-p)\rho(1-\rho),$$
$$v_d = (1-p)\rho\left(1-\frac{\rho^2}{2}\right), \qquad (3)$$

where v_s (v_d) is the outward velocity of the boundary for the model with the static (dynamic) symmetry and ρ is the walker density. The first term comes from the branching process of a walker at the boundary and the second term comes from that near the boundary. The effect of hopping on the boundary velocity is omitted. We find that v_s is always smaller than v_d. This result implies that the active region grows faster in the model with the dynamic symmetry, so this model should be more active than the model with the static symmetry. This mean-field argument can be easily generalized to the models with general n offspring.

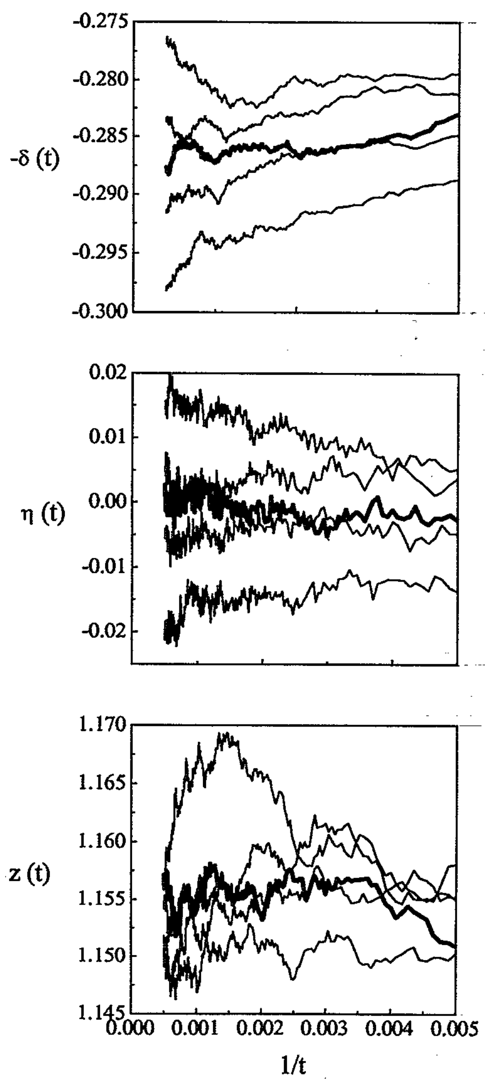

FIG. 3. Plots of the effective exponents against $1/t$. Five curves from top to bottom in each panel correspond to $p = 0.5080$, 0.5098, 0.5105, 0.5112, and 0.5140. Thick lines are critical lines ($p = 0.5105$).

As the critical probability depends enormously on the symmetry of branching processes (especially $n = 2$), it is meaningless to ask whether p_c monotonically increases with n for the ordinary BAW models. These models have not been classified by the symmetry of branching processes. The branching process of the ordinary BAW model with one offspring basically belongs to the process with the dynamic symmetry. If we compare the critical probabilities of the BAW models with dynamic branching only, we expect that p_c will monotonically increase with n; i.e., the dynamic branching process always makes the system more active. In the next section, we study the BAW model with one and two offspring created by dynamic branching and examine whether the reentrant behavior seen in the case of static branching (Sec. II) disappears.

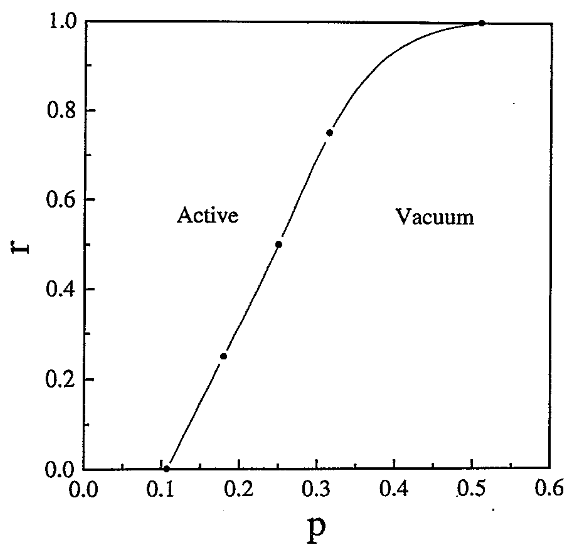

FIG. 4. r-p phase diagram for the BAW model with one and two offspring with relative ratio (dynamic branching). The full line between the Monte Carlo data is a guide to the eye.

IV. THE BAW MODEL WITH ONE AND TWO OFFSPRING: DYNAMIC BRANCHING

We consider the BAW model with one and two offspring branched dynamically with a relative ratio. The evolution rules of this model are exactly the same as in Sec. II except that static branching is replaced with dynamic branching for the two-offspring branching process.

We perform dynamic Monte Carlo simulations for $r = 0.25$, 0.50, and 0.75. Estimated values of p_c are listed in Table II. As expected, p_c increases monotonically as r becomes larger (more offsprings are created). The reentrant phase transition disappears entirely in this model with dynamic branching (see Fig. 4). Of course, the critical behavior at the absorbing transitions belongs to the DP universality class for $0 \leq r < 1$.

TABLE II. Critical hopping probability p_c for the BAW model with one and two offspring created by the dynamic branching. r is the relative probability of the two-offspring process. Numbers in parentheses represent the errors in the last digits.

r	0	0.25	0.50	0.75	1
p_c	0.1070(5)	0.18(1)	0.25(1)	0.315(5)	0.5105(7)

V. SUMMARY

We study the BAW models with static (ordinary) branching and dynamic branching. With the static branching, the BAW model with one and two offspring shows a reentrant phase transition from vacuum to an active state and finally into vacuum again as the relative rate of the two-offspring process increases. We argue that this reentrant property originates from the static reflection symmetry of the two-offspring branching process.

The ordinary BAW model with two offspring does not have an active phase at finite values of hopping probability. We introduce the BAW model with two offspring created by dynamic branching and show that this model exhibits a continuous phase transition from an active phase into vacuum at finite hopping probability. Its critical behavior belongs to the same non-DP universality class as in the ordinary BAW model with four offspring. We also study the BAW model with one and two offspring created by dynamic branching. The reentrant behavior disappears and our conventional wisdom is recovered, as expected. Our results shed light on how the system can be active by different branching processes and the BAW model with dynamic branching may serve as another simple model exhibiting the non-DP critical behavior.

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