Exactly solvable two-terminal heat engine with asymmetric Onsager coefficients: Origin of the power-efficiency bound

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An engine producing a finite power at the ideal (Carnot) efficiency is a dream engine which is not prohibited by the thermodynamic second law. Some years ago, a two-terminal heat engine with *asymmetric* Onsager coefficients in the linear response regime was suggested by Benenti *et al.* [Phys. Rev. Lett. **106**, 230602 (2011)], as a prototypical system to make such a dream come true with nondivergent system parameter values. However, such a system has never been realized, in spite of many trials. Here, we introduce an exactly solvable two-terminal Brownian heat engine with the asymmetric Onsager coefficients in the presence of a Lorenz (magnetic) force. Nevertheless, we show that the dream engine regime cannot be accessible, even with the asymmetric Onsager coefficients, due to an instability keeping the engine from reaching its steady state. This is consistent with recent tradeoff relations between the engine power and efficiency, where the (cyclic) steady-state condition is implicitly presumed. We conclude that the inaccessibility to the dream engine originates from the steady-state constraint on the engine.

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I. INTRODUCTION

Is it possible to attain the theoretically maximum efficiency, i.e., the Carnot efficiency η_C, at a finite power? As well known from textbooks [1], η_C is attainable in a reversible or quasistatic process. However, the power of such a reversible engine vanishes, as it takes an infinite time to complete one engine cycle. If we operate the engine in a finite-time cycle, we can have a finite power but usually with irreversible heat dissipation; thus the efficiency should be lower than η_C. This is why there has been a widespread belief that the *dream* engine is impossible, i.e., it is impossible to achieve η_C and a finite power simultaneously, even though there has been no rigorous proof for a long time.

In this context, the recent claim by Benenti, Saito, and Casati (BSC) [2] was surprising. They showed in the framework of the linear irreversible thermodynamics that the dream engine is possible in a two-terminal thermoelectric device in the presence of a magnetic field breaking the microscopic irreversibility. They considered a thermodynamic system where two currents J_1 and J_2 are generated by two thermodynamic forces X_1 and X_2 in the linear response regime as follows:

$$J_1(\mathbf{B}) = L_{11}(\mathbf{B})X_1 + L_{12}(\mathbf{B})X_2,$$
$$J_2(\mathbf{B}) = L_{21}(\mathbf{B})X_1 + L_{22}(\mathbf{B})X_2, \quad (1)$$

where L_{ij} is an element of the Onsager matrix L and a function of the magnetic field $\mathbf{B}$. In the case of $\mathbf{B} = 0$, the Onsager matrix is proven to be symmetric due to the microreversibility or the detailed balance [3]. However, it can be *asymmetric* with nonzero $\mathbf{B}$, only satisfying the Onsager-Casimir relation [4] as $\mathsf{L}(\mathbf{B}) = \mathsf{L}^{\mathsf{T}}(-\mathbf{B})$, with T denoting the transpose. We note that the fluctuation-dissipation relation is still satisfied with nonzero $\mathbf{B}$, while the Onsager symmetry is broken [5].

BSC [2] showed that Carnot efficiency at a finite power is attainable when the following conditions are satisfied:

$$\mathcal{L} \equiv 4 \det \mathsf{L} - (L_{12} - L_{21})^2 = 0 \quad \text{and} \quad \left| s \equiv \frac{L_{12}}{L_{21}} \right| > 1, \quad (2)$$

where det denotes the determinant, and s is called the symmetry factor. The first equation represents the maximum efficiency condition for given s. This result is presented in Fig. 1 as the solid curve, which is the curve of the maximum efficiency as a function of s constrained by the thermodynamic second law. One can see that η_C is accessible for $|s| \geqslant 1$, where the power (proportional to $s^2 - 1$) is finite except for the symmetric case ($s = 1$). This suggests that the dream engine could be possible with a symmetry breaking induced by the magnetic field.

This study triggered a flurry of subsequent discussions on developing engine mechanisms achieving the Carnot efficiency at a finite power or in an irreversible process [6–16]. From these studies, several mechanisms have been suggested to realize the dream engine, for example, by approaching the criticality of the engine system [9], an infinitely fast process [11], and cycling in the diverging damping coefficient (or vanishing-relaxation-time) limit [12]. More importantly, several tradeoff relations between the power and the efficiency have been found for various situations [10,13,14,17], such as

$$\mathcal{P} \leqslant \Theta(\eta_C - \eta), \quad (3)$$

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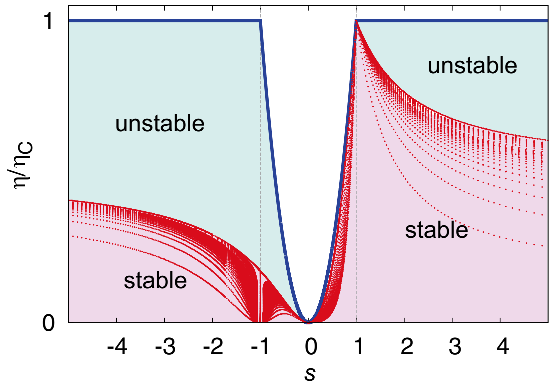

FIG. 1. Efficiency as a function of s. The (blue) solid curve is the maximum efficiency obtained by Benenti *et al.* [2], the region below which is allowed by the thermodynamic second law. Scattered (red) points denote the calculated maximum efficiencies of our model at various parameter values subject to the stable steady-state condition. The blueish region above the scattered points is unstable in our model.

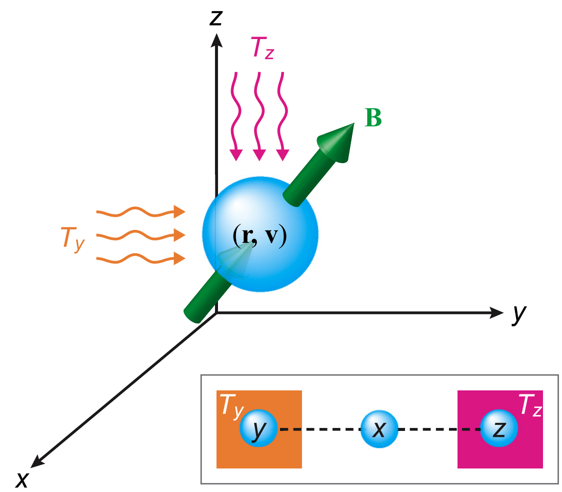

FIG. 2. The two-terminal Brownian engine in the three-dimensional space. (Inset) This model can be interpreted as a three-particle system in the one-dimensional space with one particle outside of the heat reservoirs.

where $\mathcal{P}$ is the power, η is the efficiency, and Θ is a system-dependent positive constant. This relation sets a constraint that the power should vanish to attain η_C unless Θ diverges. All these findings strongly assert that some diverging limits are necessary to attain the dream engine.

On the other hand, the BSC formulation [2] does not require any divergence of parameters for achieving η_C at a finite power. In other words, if we have the model described by Eq. (1) with $s \neq 1$ and find a set of parameters with moderate values satisfying Eq. (2), the dream engine should be realized. In this sense, the BSC theory [2] and all the subsequent studies look contradictory. Therefore, it is important to study a concrete two-terminal model with asymmetric Onsager coefficients for investigating the possibility of attaining the Carnot efficiency at a finite power in a realistic situation with moderate parameters.

However, nobody has succeeded in finding such a two-terminal engine with $s \neq 1$. In a purely coherent two-terminal system, for example, the off-diagonal elements of the Onsager matrix turn out to be even functions of the magnetic field; thus, they are always symmetric and no *reversible* currents responsible for the dream engine are possible [6,18]. Inelastic scatterings and interactions are suggested to break the symmetry, but no explicit cases are reported. To detour this problem, some studies introduced a third terminal (or more terminals) with a specific condition for mimicking a two-terminal engine [6,19], a time-averaged Onsager matrix for a periodically driven system [8], and the Nernst effect [20]. However, they are not exactly matched to the two-terminal system described by Eq. (1), and no dream engine was realized in the steady state.

In this study we introduce an exactly solvable stochastic model which manifests the symmetry breaking of the Onsager matrix in the presence of a magnetic field. We find that many sets of parameters with moderate values satisfy the dream engine condition in Eq. (2). Nevertheless, this does not guarantee the existence of the dream engine alone, because one should check the stability of the steady state for such a set of parameters. It turns out that there is no stable steady state in all those sets of parameters satisfying the dream engine condition. Our finding stresses the importance of the *boundary condition* or *intrinsic constraint* imposed for an engine problem, which is the steady-state or periodic-cycle condition, inevitably required for steady production of work from an engine. We conclude that this constraint plays the most crucial role in forbidding the dream engine realized, rather than the symmetry breaking of the Onsager matrix, which is a necessary condition.

II. MODEL

We consider an underdamped Brownian dynamics of a charged particle with mass m in the three-dimensional space as illustrated in Fig. 2. Its position and velocity are denoted by $\mathbf{r} = (x, y, z)^{\mathsf{T}}$ and $\mathbf{v} = (v_x, v_y, v_z)^{\mathsf{T}}$, respectively. The particle moves in a magnetic field $\mathbf{B} = (B_x, B_y, B_z)^{\mathsf{T}}$ and is confined in a harmonic potential with stiffness $k(> 0)$. Its dynamics along the y and z axis are affected by heat reservoirs with different temperatures T_y and T_z, respectively, while the dynamics along the x axis is not affected by any heat reservoir and is thus deterministic [21]. A linear external nonconservative force (torque), $\mathbf{f}_{\rm nc} = \epsilon y\hat{x} + \delta x\hat{y}$, is applied to extract work out of the engine.

The Langevin equation for this particle can be written as

$$\mathbf{v} = \dot{\mathbf{r}},\ m\dot{\mathbf{v}} = -k\mathbf{r} + \mathsf{F}_{\rm nc}\mathbf{r} + \mathbf{v}\times\mathbf{B} - \Gamma\mathbf{v} + \boldsymbol{\xi} \quad \text{with}$$

$$\mathsf{F}_{\rm nc} = \begin{pmatrix} 0 & \epsilon & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ \Gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \text{and} \quad \boldsymbol{\xi} = \begin{pmatrix} 0 \\ \xi_y \\ \xi_z \end{pmatrix}, \tag{4}$$

where $\xi_i \ (i = y, z)$ is a white Gaussian noise satisfying $\langle \xi_i(t)\xi_j(t')\rangle = 2\gamma T_i\delta_{ij}\delta(t-t')$ in the Boltzmann unit ($k_B = 1$) and $\mathbf{v}\times\mathbf{B}$ is the Lorentz force. Note that $\mathbf{f}_{\rm nc}(= \mathsf{F}_{\rm nc}\mathbf{r})$ becomes conservative when $\epsilon = \delta$, is otherwise nonconservative, then drives the system out of equilibrium. In addition, the temperature difference between T_y and T_z is another driving force. Thus, there are two thermodynamic forces driving the system into a nonequilibrium state, such as

$$X_1 \equiv \delta - \epsilon \quad \text{and} \quad X_2 \equiv 1/T_y - 1/T_z \ (T_y < T_z). \tag{5}$$

Note that the Carnot efficiency is given as $\eta_C = T_y X_2$.

The two-dimensional version has been studied in various contexts with and without a magnetic field [22–27], and the Onsager coefficients turn out to be symmetric even in the presence of a magnetic field (see Appendix A). This is why we resort to a more complicated three-dimensional version, still keeping only two terminals. Equation (4) can be also interpreted as a three-particle system in the one-dimensional space, each of which is confined in a harmonic potential and interacts to each other through $\mathbf{f}_{\rm nc}$ and the Lorenz force, as illustrated in inset of Fig. 2. Two particles are in contact with two different heat reservoirs, respectively, and the remaining one particle is outside of the reservoirs. The two-dimensional version does not carry this extra particle with the y-z exchange (left-right) symmetry.

In our model, we calculate the heat transferred from the i-axis reservoir into the particle $Q_i(t)$ and the work extraction due to the nonconservative force $W(t)$ by the standard stochastic energetics [23,28]. During an infinitesimal time interval $[t, t+dt]$, their incrementals can be written as

$$dQ_i(t) = v_i(t) \circ [-\gamma v_i(t)dt + d\Xi_i(t)], \tag{6}$$

$$dW(t) = -\mathbf{f}_{\rm nc} \cdot d\mathbf{r} = -[\epsilon v_x(t)y(t) + \delta x(t)v_y(t)]dt, \tag{7}$$

where $\circ$ denotes the Stratonovich multiplication [28] and $d\Xi_i(t) \equiv \int_t^{t+dt} dt' \xi_i(t')$, satisfying $\langle d\Xi_i(t)\rangle = 0$ and $\langle d\Xi_i(t) d\Xi_j(t)\rangle = 2\gamma T_i \delta_{ij} dt$. From the thermodynamic first law, $dE(t) = dQ_y(t) + dQ_z(t) - dW(t)$, where $dE(t)$ is the internal energy change during $[t, t+dt]$. We consider the steady-state average only, denoted by $\langle\cdots\rangle_{\rm s}$. As $\langle dE\rangle_{\rm s} = 0$, we have two independent energy currents. From the Stratonovich algebra, $\langle v_i \circ d\Xi_i(t)\rangle_{\rm s} = \gamma T_i dt/m$, the rates of the heat and work are given by

$$q_i \equiv \langle \dot{Q}_i\rangle_{\rm s} = \frac{\gamma}{m}(T_i - m\langle v_i^2\rangle_{\rm s}), \tag{8}$$

$$\mathcal{P} \equiv \langle P\rangle_{\rm s} = (\epsilon - \delta)\langle x v_y\rangle_{\rm s}, \tag{9}$$

where $\dot{Q}_i = dQ_i/dt$, $P = dW/dt$, and the second equation is obtained by using the steady-state property as $\frac{d}{dt}\langle x(t)y(t)\rangle_{\rm s} = \langle v_x(t)y(t)\rangle_{\rm s} + \langle x(t)v_y(t)\rangle_{\rm s} = 0$.

III. ONSAGER COEFFICIENTS

We define two currents J_1 and J_2 as follows:

$$J_1 \equiv \frac{\langle x v_y\rangle_{\rm s}}{T_y}, \ J_2 \equiv q_z, \tag{10}$$

where q_z is the heat current out of the high-temperature reservoir and the work current (power) is given by $\mathcal{P} = -J_1 X_1 T_y$, as in the standard linear irreversible thermodynamics [3]. Then, the total entropy production (EP) rate $\langle \dot{S}_{\rm tot}\rangle_{\rm s}$ can be written as

$$\langle \dot{S}_{\rm tot}\rangle_{\rm s} = -\frac{q_y}{T_y} - \frac{q_z}{T_z} = J_1 X_1 + J_2 X_2, \tag{11}$$

and the thermodynamic second law puts a constraint on the Onsager matrix as

$$\mathcal{L} = 4\det \mathsf{L} - (L_{12} - L_{21})^2 \geqslant 0 \ \text{for} \ L_{11}, L_{22} > 0. \tag{12}$$

Note that in the so-called *tight-coupling* case with $\det \mathsf{L} = 0$ [29], the Onsager symmetry ($s = 1$) is required by the above constraint.

We now calculate J_1 and J_2 explicitly, i.e., $\langle x v_y\rangle_{\rm s}$ and $\langle v_z^2\rangle_{\rm s}$, by following the standard procedure for solving a multivariate Ornstein-Uhlenbeck process [30,31]. Introduce a state vector $\mathbf{z} \equiv (x, y, z, v_x, v_y, v_z)^{\mathsf{T}}$ and a noise vector $d\Xi(t) \equiv (d\Xi_1(t), d\Xi_2(t), \ldots, d\Xi_6(t))^{\mathsf{T}}$ with $\langle d\Xi(t)\rangle = 0$ and $\langle d\Xi(t) d\Xi^{\mathsf{T}}(t)\rangle = 2\mathsf{D}dt$, where D is a 6×6 symmetric diffusion matrix. Then the equation of motion, Eq. (4), can be written in the form of the Ornstein-Uhlenbeck process as

$$d\mathbf{z} = -\mathsf{A}\mathbf{z}dt + d\Xi, \tag{13}$$

where

$$\mathsf{A} = \frac{1}{m}\begin{pmatrix} 0 & 0 & 0 & -m & 0 & 0 \\ 0 & 0 & 0 & 0 & -m & 0 \\ 0 & 0 & 0 & 0 & 0 & -m \\ k & -\epsilon & 0 & 0 & -B_z & B_y \\ -\delta & k & 0 & B_z & \gamma & -B_x \\ 0 & 0 & k & -B_y & B_x & \gamma \end{pmatrix}, \tag{14}$$

and $\mathsf{D}_{ij} = 0$ for all elements except $\mathsf{D}_{55} = \gamma T_y/m^2$ and $\mathsf{D}_{66} = \gamma T_z/m^2$.

The covariant matrix Σ is defined as $\Sigma \equiv \langle \mathbf{z}\mathbf{z}^{\mathsf{T}}\rangle_{\rm s} = \Sigma^{\mathsf{T}}$, which satisfies

$$\mathsf{A}\Sigma + \Sigma\mathsf{A}^{\mathsf{T}} = 2\mathsf{D} \tag{15}$$

from the steady-state condition $d\Sigma = 0$ [30,31]. It is straightforward to solve Eq. (15), in general, but its solution for Σ is quite complicated. In order to calculate the Onsager coefficients in Eq. (1), it is convenient to employ a perturbation expansion near the steady state (equilibrium) when $\delta = \epsilon$ and $T_z = T_y$, instead. Up to the lowest order in the thermodynamic forces X_1 and X_2 in Eq. (5), we expand the matrices as

$$\begin{aligned} &\mathsf{A} = \mathsf{A}_0 + \mathsf{A}_1 X_1, \ \ \mathsf{D} = \mathsf{D}_0 + \mathsf{D}_2 X_2, \\ &\Sigma = \Sigma_0 + \Sigma_1 X_1 + \Sigma_2 X_2, \end{aligned} \tag{16}$$

where the unperturbed ones $\mathsf{A}_0 = \mathsf{A}|_{\delta=\epsilon}$ and $\mathsf{D}_0 = \mathsf{D}|_{T_z=T_y}$, and the fist-order corrections $[\mathsf{A}_1]_{ij} = 0$ except $[\mathsf{A}_1]_{51} = -1/m$ and $[\mathsf{D}_2]_{ij} = 0$ except $[\mathsf{D}_2]_{66} = \gamma T_y^2/m^2$ for all i and j.

The covariant matrix expansion with Σ_0, Σ_1, and Σ_2 can be obtained by a series of equations derived from Eq. (15) as

$$\begin{aligned} &\mathsf{A}_0\Sigma_0 + \Sigma_0\mathsf{A}_0^{\mathsf{T}} = 2\mathsf{D}_0, \\ &\mathsf{A}_0\Sigma_1 + \Sigma_1\mathsf{A}_0^{\mathsf{T}} = -\mathsf{A}_1\Sigma_0 - \Sigma_0\mathsf{A}_1^{\mathsf{T}}, \\ &\mathsf{A}_0\Sigma_2 + \Sigma_2\mathsf{A}_0^{\mathsf{T}} = 2\mathsf{D}_2. \end{aligned} \tag{17}$$

First, we find

$$\Sigma_0 = T_y \begin{pmatrix} k/K & \epsilon/K & 0 & 0 & 0 & 0 \\ \epsilon/K & k/K & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/k & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/m & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/m \end{pmatrix}, \quad (18)$$

with $K = k^2 - \epsilon^2$. The stability of the unperturbed steady state is guaranteed by the positivity of all eigenvalues [30,31], which gives

$$K = k^2 - \epsilon^2 > 0 \text{ (stability condition)}. \quad (19)$$

We can also find Σ_1 and Σ_2 from Eqs. (17) and (18).

From Eqs. (1) and (10), we express the Onsager matrix L by the elements of the covariant matrix Σ as

$$\mathsf{L} = \begin{pmatrix} [\Sigma_1]_{15}/T_y & [\Sigma_2]_{15}/T_y \\ -\gamma[\Sigma_1]_{66} & \gamma\left(T_y^2/m - [\Sigma_2]_{66}\right) \end{pmatrix}. \quad (20)$$

For simplicity, we set $B_x = 0$ as an example. Then we get

$$\begin{aligned} L_{11} &= \frac{1}{\gamma \mathcal{G}}\left[(2k^2 - \epsilon^2)\gamma^2 C_0 C_2 + k B_z^2 (C_1 + m\epsilon^2) C_2 \right. \\ &\quad \left. + m\epsilon^2\left(m\epsilon^2 B_z^2 + 2k\gamma^2 B_y^2\right)\right], \\ L_{22} &= \frac{\gamma T_y^2 B_y^2}{m\mathcal{G}}\left[(2k^2 - \epsilon^2)\left(2\gamma^2 C_3 + m\epsilon^2 B_y^2\right) \right. \\ &\quad \left. + 2k\left(C_3^2 + k^2 B_y^2 B_z^2\right)\right], \\ L_{12} &= \frac{\epsilon T_y B_y^2}{\mathcal{G}}[(2k^2 - \epsilon^2)\gamma C_2 + 2k\gamma m\epsilon^2 - \epsilon B_z C_1], \\ L_{21} &= \frac{\epsilon T_y B_y^2}{\mathcal{G}}[(2k^2 - \epsilon^2)\gamma C_2 + 2k\gamma m\epsilon^2 + \epsilon B_z C_1], \end{aligned} \quad (21)$$

where C_0, C_1, C_2, C_3, and $\mathcal{G}$ are given as

$$\begin{aligned} &C_0 = B_y^2 + B_z^2, C_1 = kC_0 + m\epsilon^2, C_2 = C_0 + 2\gamma^2, \\ &C_3 = kB_z^2 + m\epsilon^2, \\ &\mathcal{G} = [\{(2k^2 - \epsilon^2)\gamma^2 + k(C_1 + m\epsilon^2)\}C_2 + (m\epsilon^2)^2]C_1. \end{aligned} \quad (22)$$

Note that all C_i's ($i = 0, 1, 2, 3$) are positive and the even functions of B_y and B_z. The odd function in terms of the magnetic field appears only in the last term of the off-diagonal elements, L_{12} and L_{21}.

As expected, the Onsager-Casimir relation [3,4] is satisfied as $\mathbf{L}(\mathbf{B}) = \mathbf{L}^{\mathsf{T}}(-\mathbf{B})$, but the Onsager symmetry is broken; $\mathbf{L}(\mathbf{B}) \neq \mathbf{L}^{\mathsf{T}}(\mathbf{B})$, seen in Eq. (21). In contrast to the two-dimensional case, we find indeed a two-terminal model with the asymmetric Onsager matrix, i.e., $s \neq 1$ for the three-dimensional version.

It is interesting to note that the Onsager matrix becomes symmetric ($s = 1$) when $B_z = 0$ with $B_y \neq 0$ in Eq. (21). Moreover, $\mathcal{L} = 4\det(\mathsf{L}) = 0$ (tight coupling), implying that the reversible process is possible with $\langle \dot{S}_{\rm tot}\rangle_{\rm s} = 0$ in Eq. (11) at $X_1 = -\epsilon T_y X_2$ and thus the efficiency η can reach the Carnot efficiency $\eta_{\rm C}$.

IV. EFFICIENCY, POWER, AND EP RATE

The engine efficiency η in converting the heat flowing from the high-temperature reservoir into the power is defined as

$$\eta = \frac{\mathcal{P}}{q_z} = \frac{-J_1 X_1 T_y}{J_2} = \frac{-T_y X_1 (L_{11} X_1 + L_{12} X_2)}{L_{21} X_1 + L_{22} X_2}, \quad (23)$$

which is maximized for a given temperature gradient X_2 at

$$X_1 = X_1^* = -\frac{L_{22}}{L_{21}}\left(1 - \sqrt{\frac{\det \mathsf{L}}{L_{11} L_{22}}}\right) X_2, \quad (24)$$

with the maximum efficiency for given L,

$$\eta^* = \eta(X_1^*) = \eta_{\rm C} \frac{L_{11} L_{22}}{L_{21}^2}\left(1 - \sqrt{\frac{\det \mathsf{L}}{L_{11} L_{22}}}\right)^2, \quad (25)$$

where X_2 is replaced by $\eta_{\rm C} = T_y X_2$.

It is rather convenient to rewrite η^* in terms of $\mathcal{L}$ in Eq. (12) as

$$\eta^* = \frac{\eta_{\rm C}}{4}\left[\sqrt{\mathcal{Y} + (s+1)^2} - \sqrt{\mathcal{Y} + (s-1)^2}\right]^2 \ (\mathcal{Y} = \mathcal{L}/L_{21}^2), \quad (26)$$

with $\mathcal{Y} \geqslant 0$ by the thermodynamic constraint in Eq. (12). One can easily find that η^* is a monotonically decreasing function of $\mathcal{Y}$ for fixed s, so η^* can reach its highest value $\eta^{\rm max}$ at $\mathcal{Y} = 0$ as

$$\eta^{\rm max} = \begin{cases} \eta_{\rm C} & \text{for } |s| \geqslant 1 \\ s^2 \eta_{\rm C} & \text{for } |s| < 1 \end{cases}, \quad (27)$$

which is shown as the blue solid curve in Fig. 1 [2]. Note that in the symmetric case ($s = 1$), the Carnot efficiency is achieved in the tight-coupling limit ($\det \mathsf{L} = 0$).

The power and the EP rate at the maximum efficiency η^* are given as

$$\mathcal{P}^* = \mathcal{P}(X_1^*) = L_{22}\sqrt{\frac{\mathcal{Y} + (s-1)^2}{\mathcal{Y} + (s+1)^2}}\, \eta^* X_2, \quad (28)$$

$$\langle \dot{S}_{\rm tot}\rangle_{\rm s}^* = L_{22}\sqrt{\frac{\mathcal{Y} + (s-1)^2}{\mathcal{Y} + (s+1)^2}}\left(1 - \frac{\eta^*}{\eta_{\rm C}}\right) X_2^2. \quad (29)$$

Along the highest efficiency curve in Eq. (27), the power $\mathcal{P}^{\rm m}$ and the EP rate $\langle \dot{S}_{\rm tot}\rangle_{\rm s}^{\rm m}$ are obtained as

$$\mathcal{P}^{\rm m} = \frac{L_{22}\eta_{\rm C}^2}{T_y}\begin{cases} \left|\frac{s-1}{s+1}\right| & \text{for } |s| \geqslant 1 \\ s^2\left(\frac{1-s}{1+s}\right) & \text{for } |s| < 1 \end{cases}, \quad (30)$$

$$\langle \dot{S}_{\rm tot}\rangle_{\rm s}^{\rm m} = \frac{L_{22}\eta_{\rm C}^2}{T_y^2}\begin{cases} 0 & \text{for } |s| \geqslant 1 \\ (1-s)^2 & \text{for } |s| < 1 \end{cases}. \quad (31)$$

For $|s| > 1$, we find that the efficiency can reach $\eta_{\rm C}$ in Eq. (27) with nonzero power $\mathcal{P}^{\rm m}$ in Eq. (30) (dream engine) and vanishing EP in Eq. (31), which was the main result of BSC [2].

V. STABILITY

As in Eqs. (18) and (19), the unperturbed steady state (equilibrium) is stable only for $K = k^2 - \epsilon^2 > 0$. Thus, we should examine the results of the last section within the

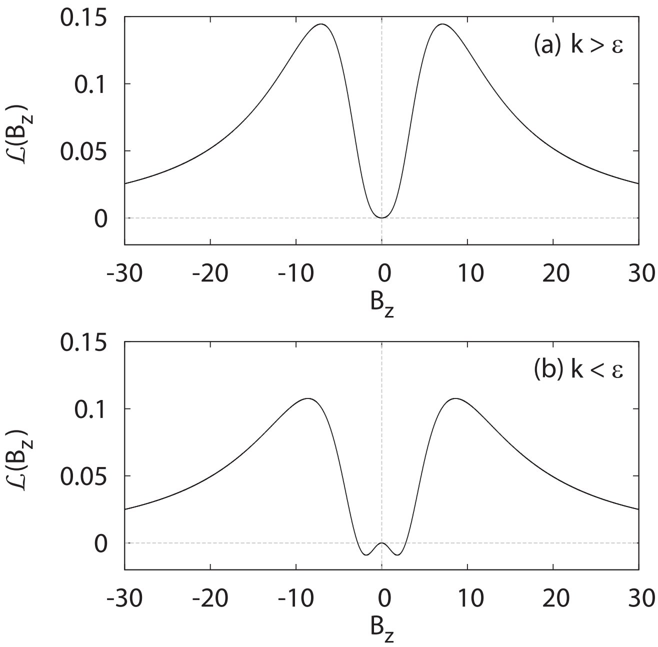

FIG. 3. Plots of $\mathcal{L}$ as a function of B_z when (a) $k > |\epsilon|$ and (b) $0 < k < |\epsilon|$, respectively. For both cases, $\mathcal{L} = 0$ at $B_z = 0$ and $\mathcal{L} \to 0^+$ in $|B_z| \to \infty$ limit.

stability condition. It is easy to see that $L_{11}, L_{22} > 0$ and $\mathcal{G} > 0$ for $k^2 > \epsilon^2$ in Eqs. (21) and (22). We need to check whether the dream engine condition, i.e., $\mathcal{L} = 0$ for $|s| > 1$ in Eq. (2), is possible for $k^2 > \epsilon^2$.

We rewrite $\mathcal{L}$ explicitly, using Eq. (21), as

$$\mathcal{L} = \frac{4T_y^2 B_y^2}{m\mathcal{G}^2}\left\{[l_{11}][l_{22}] - m\epsilon^2 B_y^2 \gamma^2 [(2k^2 - \epsilon^2)C_2 + 2km\epsilon^2]^2\right\}, \tag{32}$$

where $[l_{11}]$ and $[l_{22}]$ are the expressions inside of the $[\cdot\cdot]$ of L_{11} and L_{22}, respectively, in Eq. (21). First, $\mathcal{L} = 0$ and $s = 1$ for $B_z = 0$ pointed out in Sec. III. Second, $\mathcal{L}$ is the even function of B_z. Thus, $\mathcal{L}$ can be written in a power series of B_z^2 as

$$\mathcal{L} = \frac{4T_y^2 B_y^2}{m\mathcal{G}^2} \sum_{n=1}^{5} a_{2n} B_z^{2n}, \tag{33}$$

where the coefficient a_{2n} is a function of m, k, ϵ^2, γ^2, and B_y^2.

It is straightforward to prove that all coefficients a_{2n}'s are definitely positive for $k^2 > \epsilon^2$ (not shown here), which implies that the $\mathcal{L} = 0$ condition is satisfied only at $B_z = 0$, and thus $s = 1$. Thus, the dream engine cannot be achieved for any set of parameters compatible with the stability condition. In Fig. 3, $\mathcal{L}$ versus B_z is plotted for a typical parameter set when (a) $k > |\epsilon|$ and (b) $0 < k < |\epsilon|$. Note that $\mathcal{L}$ can vanish at a nonzero B_z only in the unstable case (b). Our result for this exactly solvable model clearly shows the key role of the intrinsically imposed constraint, i.e., the existence of a stable steady state in an engine problem.

We numerically check the maximum efficiency values in the stable region. As η^* is the monotonically decreasing function of $\mathcal{Y}$ for a given s in Eq. (26), the highest possible efficiency value can be obtained at the smallest possible $\mathcal{Y}$, subject to the stability condition ($k > |\epsilon|$).

For this calculation, we vary k ($0 \leqslant k \leqslant 7$), B_z ($-2500 \leqslant B_z \leqslant 2500$), m ($4 \leqslant m \leqslant 10^6$), γ ($0.01 \leqslant \gamma \leqslant 1$), $100 \leqslant B_y \leqslant 10^6$, and $1 \leqslant T_2 \leqslant 10^6$ with fixed parameter $B_x = 0$. The results are presented in Fig. 1, where the stable region does not reach the Carnot efficiency line except at $s = 1$. Note that the stable region is much smaller for negative s and in particular, does not exist for $s = -1$. This is special in our model with $B_x = 0$, which can be easily noticed in Eq. (21), i.e., $L_{12} + L_{21} \propto (2k^2 - \epsilon^2)C_1 + 2km\epsilon^2$ can never be zero for $k^2 > \epsilon^2$.

VI. SUMMARY AND DISCUSSION

In summary, we explicitly showed in an exactly solvable model that the stability constraint for the steady state is crucial in prohibiting a dream engine. The asymmetry of the Onsager matrix L may arise in a two-terminal engine, but the reversible limit for a dream engine cannot be accessible due to the stability condition of the unperturbed steady state.

The power-efficiency tradeoff relation derived by Dechant and Sasa (DS) [13] should be applied to our model, which includes a nonconservative force in the framework of an underdamped dynamics. The DS derivation is based on the entropic bound on general irreversible currents, which is written as

$$\langle \dot{Q}_i \rangle^2 \leqslant \zeta_i \langle \dot{S}_{\text{tot}} \rangle, \tag{34}$$

where $\langle \cdots \rangle$ denotes the ensemble average at an arbitrary time t, a time-dependent coefficient $\zeta_i = \gamma T_i \langle v_i^2 \rangle$, and $\langle \dot{S}_{\text{tot}} \rangle = \langle \dot{S}_{\text{sys}} \rangle - \langle \dot{Q}_y \rangle / T_y - \langle \dot{Q}_z \rangle / T_z$ with the Shannon entropy change rate $\langle \dot{S}_{\text{sys}} \rangle$. Note that this entropic bound is valid even with the Lorentz force. Then, we can show that the instantaneous power

$$\langle P \rangle \leqslant \frac{\zeta_z \eta}{T_y}\left[\eta_{\text{C}} - \eta + \frac{T_y \langle \dot{S}_{\text{sys}} \rangle - \langle \dot{E} \rangle}{\langle \dot{Q}_z \rangle}\right], \tag{35}$$

where $\langle \dot{E} \rangle$ is the system-energy change rate. In the steady state with $\langle \dot{S}_{\text{sys}} \rangle = \langle \dot{E} \rangle = 0$, Eq. (35) returns back to Eq. (3). If the system is in a transient state, the power may not vanish at $\eta = \eta_{\text{C}}$ in general. This clearly shows the importance of the steady-state constraint for the power-efficiency bound. The above discussion can be extended to a cyclic engine. The similar bound as in Eq. (3) can be derived in a cyclic steady state [13], where the Shannon entropy change of the system over one cycle is zero. In Appendix B, the detailed derivation for the work extraction per cycle is given for a cyclic engine.

In conclusion, we show that the steady-state constraint is the key ingredient keeping a dream engine from being realized, rather than the asymmetry of the Onsager matrix. Thus, the BSC claim [2] based on the Onsager asymmetry should be understood as a misleading result caused by overlooking the importance of the intrinsically imposed boundary condition.

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APPENDIX A: TWO-DIMENSIONAL CASE

Consider the equation of motion, Eq. (4), in the two-dimensional space with

$$\mathsf{F}_{\rm nc} = \begin{pmatrix} 0 & \epsilon \\ \delta & 0 \end{pmatrix}, \ \Gamma = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \text{ and } \xi = \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix}, \quad \text{(A1)}$$

where ξ_i $(i = x, y)$ is a white Gaussian noise satisfying $\langle \xi_i(t)\xi_i(t')\rangle = 2\gamma T_i \delta(t - t')$ in the Boltzmann unit and $\mathbf{B} = B\hat{z}$ in the z direction. The thermodynamic forces are defined as

$$X_1 \equiv \delta - \epsilon \quad \text{and} \quad X_2 \equiv 1/T_x - 1/T_y \quad (T_x < T_y), \quad \text{(A2)}$$

and the currents are

$$J_1 \equiv \frac{\langle x v_y\rangle_{\rm s}}{T_x}, \ J_2 \equiv q_y = \frac{\gamma}{m}\left(T_y - m\langle v_y^2\rangle_{\rm s}\right), \quad \text{(A3)}$$

where q_y is the heat current out of the high-temperature reservoir and the work current is given by $w = -J_1 X_1 T_x$.

In order to express the equation of motion in a multivariate Ornstein-Uhlenbeck form in Eq. (13), we introduce a state vector $\mathbf{z} = (x, y, v_x, v_y)^{\mathsf{T}}$ and a noise vector $d\Xi(t) = [d\Xi_1(t), d\Xi_2(t), d\Xi_3(t), d\Xi_4(t)]^{\mathsf{T}}$, with $\langle d\Xi(t)\rangle = 0$ and $\langle d\Xi(t) d\Xi^{\mathsf{T}}(t)\rangle = 2\mathsf{D}dt$, with

$$\mathsf{A} = \frac{1}{m}\begin{pmatrix} 0 & 0 & -m & 0 \\ 0 & 0 & 0 & -m \\ k & -\epsilon & \gamma & -B \\ -\delta & k & B & \gamma \end{pmatrix},$$

$$\mathsf{D} = \frac{1}{m^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma T_x & 0 \\ 0 & 0 & 0 & \gamma T_y \end{pmatrix}. \quad \text{(A4)}$$

The covariant matrix Σ satisfies Eq. (15) in the steady state, and its expansion near the equilibrium ($\delta = \epsilon$, $T_y = T_x$) can be obtained through Eqs. (16) and (17) with $\mathsf{A}_0 = \mathsf{A}|_{\delta=\epsilon}$, $\mathsf{D}_0 = \mathsf{D}|_{T_y = T_x}$,

$$\mathsf{A}_1 = \frac{1}{m}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{and} \quad \mathsf{D}_2 = \frac{1}{m^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma T_x^2 \end{pmatrix}. \quad \text{(A5)}$$

It is simple to find Σ_0 from Eq. (17) as

$$\Sigma_0 = T_x \begin{pmatrix} k/K & \epsilon/K & 0 & 0 \\ \epsilon/K & k/K & 0 & 0 \\ 0 & 0 & 1/m & 0 \\ 0 & 0 & 0 & 1/m \end{pmatrix}, \quad \text{(A6)}$$

with $K = k^2 - \epsilon^2$. The stability condition is given by $K > 0$. We can also find Σ_1 and Σ_2 as well.

From Eqs. (1) and (10), the Onsager matrix L is given as

$$\mathsf{L} = \begin{pmatrix} [\Sigma_1]_{14}/T_x & [\Sigma_2]_{14}/T_x \\ -\gamma[\Sigma_1]_{44} & \gamma\left(T_x^2/m - [\Sigma_2]_{44}\right) \end{pmatrix}, \quad \text{(A7)}$$

and finally we get

$$L_{11} = \frac{B^2 + \gamma^2}{\gamma C}, \ L_{22} = \frac{\gamma T_x^2 (kB^2 + m\epsilon^2)}{mC},$$
$$L_{12} = \frac{-\epsilon\gamma T_x}{C} = L_{21}, \ \text{with}\, C = 2(kB^2 + m\epsilon^2 + k\gamma^2). \quad \text{(A8)}$$

As seen in Eq. (A8), the Onsager matrix is an even function of the magnetic field B and thus is symmetric ($\mathsf{L} = \mathsf{L}^{\mathsf{T}}$, $s = 1$), like in other two-terminal particle transport systems [6,18]. We also note that $\mathcal{L} = 4\det(\mathsf{L}) = 2B^2T_x^2/(mC) > 0$ (no tight binding), implying that the reversible process ($\langle \dot{S}_{\rm tot}\rangle_{\rm s} = 0$) is impossible; thus the efficiency η cannot reach the Carnot efficiency $\eta_{\rm C}$ for nonzero B.

APPENDIX B: CYCLIC ENGINE

We consider a cyclic engine with time period τ as follows. An engine system is in contact with multiple heat reservoirs with temperature $T_i(t)$ varying periodically in time t as $T_i(t + \tau) = T_i(t)$. We assume that the system is described by a Langevin dynamics. The average heat energy $\langle Q_i\rangle$ out of the ith reservoir during one period is given by

$$\langle Q_i\rangle \equiv \int_0^\tau dt\langle \dot{Q}_i\rangle \leqslant \int_0^\tau dt|\langle \dot{Q}_i\rangle| \leqslant \int_0^\tau dt\sqrt{\zeta_i}\sqrt{\langle \dot{S}_{\rm tot}\rangle}$$
$$\leqslant \sqrt{\int_0^\tau dt\sqrt{\zeta_i}}\sqrt{\int_0^\tau dt\langle \dot{S}_{\rm tot}\rangle}, \quad \text{(B1)}$$

where Eq. (34) and the Cauchy-Schwarz inequality are applied. Then we get the inequality similar to Eq. (34) as

$$\langle Q_i\rangle^2 \leqslant \chi_i \langle \Delta S_{\rm tot}\rangle, \quad \text{(B2)}$$

with a positive constant $\chi_i = \int_0^\tau dt\sqrt{\zeta_i}$ and the total EP during one period $\langle \Delta S_{\rm tot}\rangle$.

With the two (hot and cold) reservoirs with temperatures T_h and T_c, respectively, we can easily find

$$\langle W\rangle \leqslant \frac{\chi_h \eta}{T_c}\left[\eta_{\rm C} - \eta + \frac{T_c\langle \Delta S_{\rm sys}\rangle - \langle \Delta E\rangle}{\langle Q_h\rangle}\right], \quad \text{(B3)}$$

where $\langle W\rangle$, $\langle \Delta S_{\rm sys}\rangle$, and $\langle \Delta E\rangle$ are the work production, the Shannon entropy change, and the system energy change during one period, respectively. In the cyclic steady state with $\langle \Delta S_{\rm sys}\rangle = \langle \Delta E\rangle = 0$, the work extraction is impossible at the Carnot efficiency, even though it is possible in a transient state.

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