

Continuity of the explosive percolation transition

Hyun Keun Lee,^{1,2} Beom Jun Kim,² and Hyunggyu Park³

¹*Department of Physics, University of Seoul, Seoul 130-743, Korea*

²*BK21 Physics Research Division, Sungkyunkwan University, Suwon 440-746, Korea*

³*School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea*

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The explosive percolation problem on the complete graph is investigated via extensive numerical simulations. We obtain the cluster-size distribution at the moment when the cluster size heterogeneity becomes maximum. The distribution is found to be well described by the power-law form with the decay exponent $\tau = 2.06(2)$, followed by a hump. We then use the finite-size scaling method to make all the distributions at various system sizes up to $N = 2^{37}$ collapse perfectly onto a scaling curve characterized solely by the single exponent τ . We also observe that the instant of that collapse converges to a well-defined percolation threshold from below as $N \rightarrow \infty$. Based on these observations, we show that the explosive percolation transition in the model should be continuous, contrary to the widely spread belief of its discontinuity.

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The term *explosive percolation* was proposed in Ref. [1] to describe a sudden appearance of a macroscopic cluster in a network growth model with the so-called product rule considered on the complete graph. This growth rule, an example of an Achlioptas process (AP), is then studied on the two-dimensional lattice [2,3] and on the scale-free networks [4–6] as well, yielding similar results. That suddenness has been widely believed to indicate a discontinuity at the percolation transition in the thermodynamic limit [7,8], and the similar explosiveness has been observed with the other growth rules proposed later [9–14]. These observations of the explosiveness have drawn much interest due to the striking difference from the well-known continuous transition in the standard percolation models [15]. However, the discontinuity has been questioned recently and the evidence against it has accumulated [16–18]. Here, we provide a conclusive evidence for the continuity from a different perspective.

Friedman and Landsberg [9] have suggested the argument of the *powder keg* as a circumstantial description to explain the apparent discontinuity of the explosive percolation transition. Meanwhile, da Costa *et al.* [16] have reported that the explosive percolation is actually continuous for a modified version of the AP by analytically deriving the critical scaling relations based on numerical observations of power-law critical distribution of cluster size [17]. In this work, we try to unmask the (dis-)continuity in a systematic and direct way by performing a careful finite-size-scaling analysis at newly introduced *pseudo*-transition points for finite systems and show that the explosive percolation transition on the complete graph is indeed continuous in the thermodynamic limit.

The model we study is the AP with the product rule on the complete graph [1]. Start with N nodes with all links unoccupied. At each step, choose two possible unoccupied links randomly between nodes. Then, select the link merging two clusters with a smaller product of the two cluster sizes. Here, a cluster is defined as a set of nodes connected each other via occupied links. This procedure is repeated until all nodes are connected as a whole. The number of occupied links L increases one by one at each step and the occupied link density (or *time*), $t = L/N$, serves as the control parameter for

the model. The interested observable is the largest cluster size $G(t)$ which becomes macroscopic (linear in N) at sufficiently large t . The order parameter is defined as the relative size of the largest cluster, $g(t) = G(t)/N$, which remains at zero below the threshold t_c and becomes finite for $t > t_c$ in the $N = \infty$ limit.

The main question is whether the gap, $g(t)|_{t \rightarrow t_c^+}$, vanishes (continuous transition) or approaches a nonzero constant (finite jump). It may be natural to use the information above the transition point ($t > t_c$) in order to prove the (non-)existence of the gap or estimate the gap size. Thus, most of previous studies have focused on this information [1–13,16] but could not provide a definitive answer due to the extremely slow convergence of the order parameter in system size. In this work, we took the opposite approach. Using the accurate information below t_c , it is still possible to derive the upper bound for the gap, which turns out to vanish as $N \rightarrow \infty$. This guarantees the vanishing gap at the transition.

Our strategy is as follows: (i) Set up lower and upper pseudotransition points, $t_l(N)$ and $t_u(N)$, for finite size N below and above the true asymptotic percolation transition point t_c , respectively. We expect that both pseudotransition points converge to t_c as $N \rightarrow \infty$. (ii) Find the upper bound for the size increase of a largest cluster ΔG between $t_l(N)$ and $t_u(N)$. (iii) Show that this upper bound is sublinear in N , which implies no macroscopic jump of the largest cluster size through the percolation transition. This completes our argument showing the continuity at the explosive percolation transition. All procedures are done via extensive numerical simulations, typically up to $N = 2^{37} \approx 1.37 \times 10^{11}$, and the average is done over 100 ~ 5000 runs.

The most crucial step is to define the two pseudotransition points at the microscopic step level. First, we introduce the lower pseudotransition point $t_l(N)$ as the instant when the *cluster size heterogeneity* (the number of distinct cluster sizes) becomes maximum. For small t , the cluster size heterogeneity increases with t due to the randomness of clustering processes. However, the emergence of a macroscopic percolating cluster which continuously absorbs small clusters causes the heterogeneity to decrease and eventually the whole system

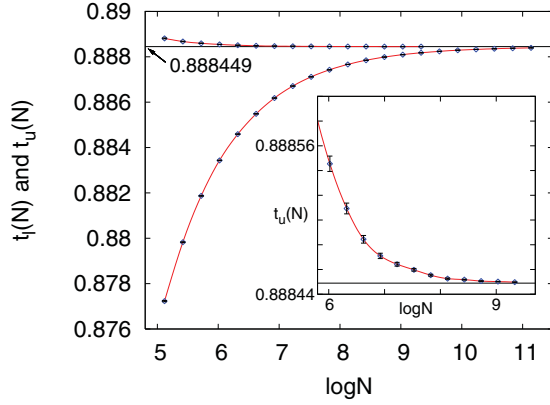


FIG. 1. (Color online) Convergence of t_l (the lower branch) and t_u (the upper one) averaged for 100 ~ 5000 different realizations for each system size ($N = 2^{17}, 2^{18}, \dots, 2^{37}$). Errors are smaller than symbol sizes, if not shown explicitly. Lines are just guides to the eyes. As N increases, t_u (t_l) decreases (increases), approaching to the well-defined value in the thermodynamic limit displayed as the horizontal line. For clarity, the upper branch is vertically enlarged in the inset.

becomes one cluster. Due to the mechanism of suppressing the emergence of large clusters, one may argue that the heterogeneity increases slowly but steadily up to just before the explosion when many different size clusters merge into one big macroscopic cluster. Thus it is reasonable to consider the *maximum heterogeneity* as a preceding symptom of the percolating onset for finite systems. In Fig. 1, the average values of $t_l(N)$ are plotted against N (lower branch), which converge to the asymptotic value of $t_c = 0.8884490(5)$ from below, as expected.

Second, we expect that the growth rate of the largest cluster also becomes maximum at the percolation transition. Microscopically, the upper pseudotransition point $t_u(N)$ is defined as one step after the moment when the second-largest cluster size becomes maximum. Thus $G(t)$ can experience a largest increase exactly when t exceeds t_u , since the second-largest cluster merges into the largest cluster. A typical growing process is displayed in Fig. 2. Note that the second-largest cluster never recovers its size after merging into the largest cluster. So there will be no explosive increase of the largest cluster size for $t > t_u(N)$. Dominance of one percolating cluster is the characteristic of the percolating phase. So it is reasonable to expect that $t_u(N)$ is just above t_c , which is consistent with numerical results (see the inset in Fig. 1). The average values of $t_u(N)$ converge to the same asymptotic value of t_c from above as $N \rightarrow \infty$.

The sample-to-sample fluctuations decrease with $N^{-0.5}$ (not shown here), which implies that both t_u and t_l are self-averaging [19], so not only the critical point but also any sample-averaged quantity are well defined in the asymptotic limit [17]. We also find numerically

$$t_u(N) - t_l(N) \sim N^{-\delta}, \quad (1)$$

with $\delta = 0.39(3)$ [20].

Now we consider $\Delta g = g(t_u) - g(t_l) = \Delta G/N$, which is the growth of the largest cluster density between two pseudotransition points through the asymptotic transition

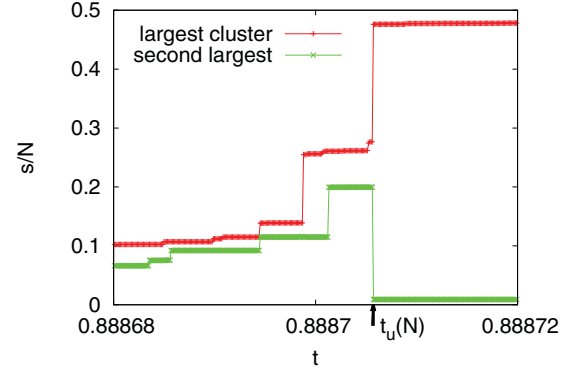


FIG. 2. (Color online) Evolutions of the largest cluster size (the upper curve) and the size of the second largest cluster (the lower curve) versus the growth step t . As t crosses t_u from below, the size of the largest cluster exhibits a sudden biggest increase since the maximum second-largest cluster merges into it at that moment. The system size is $N = 2^{23}$.

point t_c . In the thermodynamic limit, it will be the jump size (if any) of the order parameter at the percolation transition. As $g(t)$ must vanish as $N \rightarrow \infty$, we only need the information of $g(t_u)$ in principle to calculate $\Delta g|_{N \rightarrow \infty}$. Figure 3 shows the cluster-size distribution $n(s; t_u)$ of cluster size s (normalized by the total number of clusters C) at $t_u(N)$ for various different sizes N . The distribution fits extremely well with a power-law form, $n(s; t_u) \sim s^{-\tau}$ with the decay exponent $\tau = 2.06(2)$ [5,10–12] in a huge range, which is then accompanied by a little dip in the end. In Fig. 3, the largest cluster size $G(t_u)$ depending on the dip structure near the upper cutoff shows a slight trend of the sublinearity in N (moving left in the axis of s/N as N increases), which may be one symptom for the continuous transition. However, as discussed before, it cannot be conclusive even with huge system sizes studied here.

If one assumes a conventional *natural* cutoff of the power-law type distribution function, the upper cutoff which should be proportional to the largest cluster size $G(t_u)$ will scale as $N^{1/(\tau-1)} \simeq N^{0.94}$ with $\tau \simeq 2.06$. Sublinearity is estimated only by 6%, which may call for a huge system

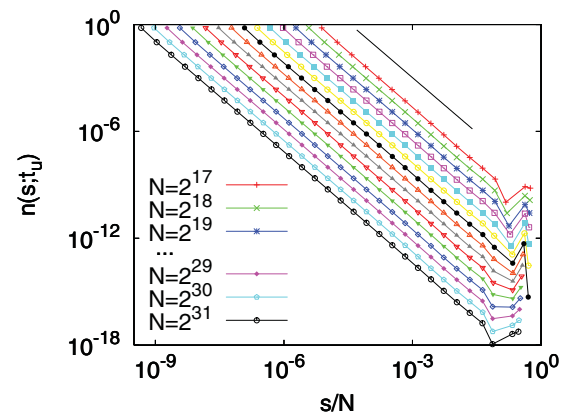


FIG. 3. (Color online) Log-binned cluster-size distributions at t_u for each N , where the horizontal axis is the cluster size s divided by N for convenience. The solid line above is a guiding line of which the decay exponent is 2.06.

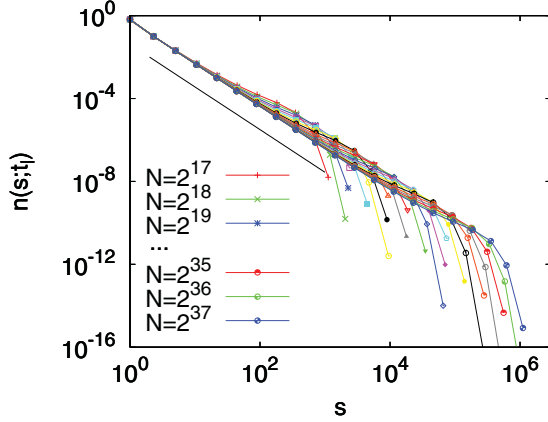


FIG. 4. (Color online) Log-binned cluster-size distributions at t_l for each N . The decay exponent of the guiding solid line below is 2.06.

size like $N \sim 10^{17}$ (beyond the present computing capability) to reach a reasonable scaling regime [$g(t_u) \lesssim 0.1$] and get any sensible extrapolation to the thermodynamic limit. Most of previous studies [1–13,16] basically depend on the data in this supercritical regime ($t > t_c$). Nevertheless, the scaling plot with this natural cutoff shows a *reasonable* collapse including the dip structure at the end but involving big statistical errors (not shown here).

In efforts to find conclusive evidences, we scrutinize the cluster-size distribution $n(s; t_l)$ at $t_l(N)$, which shows again the power-law decay with the same decay exponent $\tau = 2.06(2)$ followed by a hump near the upper cutoff (see Fig. 4). It has a much shorter (but still quite broad) power-law regime but exhibits much better statistics even in the hump region (see Fig 5).

In contrast to the cluster distribution at $t_u(N)$, $n(s; t_l)$ shows a fast exponential decay near the cutoff s_f . This sharp cutoff originates from the nature of the growth (product) rule which discourages the merging of bigger clusters before explosion. One may estimate $s_f(N)$ at t_l as follows. It is appropriate to

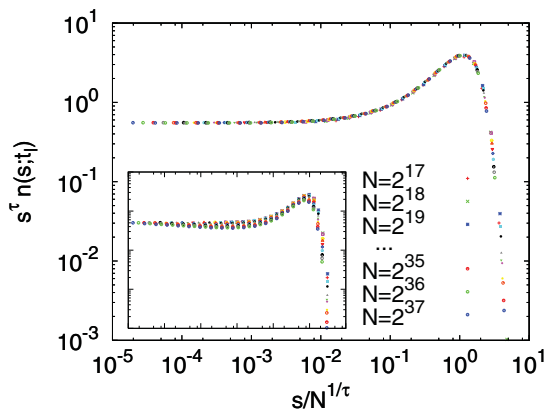


FIG. 5. (Color online) Finite-size scaling of the cluster size distribution $n(s)$ at $t_l(N)$: All data points plotted in the form of $s^\tau n(s)$ versus $sN^{-1/\tau}$ at $\tau = 2.06$ collapse into a single curve. (Inset) The same plot but with $\tau = 2$. This clearly shows that $\tau \approx 2.06$ (> 2) gives us the much better quality of the FSS collapse.

estimate the upper cutoff s_f by assuming the $O(1)$ number of clusters left beyond the cutoff, i.e., $\sum_{s \geq s_f} n(s) \sim 1/C$, where C is the total number of clusters in the system. Note that C scales linearly with N [21]. Since $n(s)$ decays exponentially fast (or faster) near s_f , $\sum_{s \geq s_f} n(s) \approx (\Delta s_f) n(s_f)$ with a finite characteristic scale Δs_f of the fast decaying part. Consequently, one can find $n(s_f) \sim s_f^{-\tau} \sim 1/C \sim 1/N$, which leads to

$$s_f(N) \sim N^{1/\tau}, \quad (2)$$

at $t = t_l(N)$. Note that this cutoff scales differently from the natural cutoff. The huge difference in the largest cluster size $G(t)$ just below and above t_c leads to its abrupt and explosive increase through the percolation transition, which is the main difference between the ordinary and explosive percolation. However, the magnitude of the explosion may be still sublinear in N as discussed before.

The validity of Eq. (2) can be checked numerically through the finite-size-scaling (FSS) analysis for the entire distribution function $n(s; t_l)$. Assuming a *single* characteristic cluster size (proportional to s_f), one can write a FSS form for the distribution $n(s; t_l)$ as

$$n(s; t_l) = s^{-\tau} f(s/s_f) = s^{-\tau} f(sN^{-1/\tau}), \quad (3)$$

where $f(x)$ is the scaling function that becomes an $O(1)$ constant for $x \ll 1$ and decays exponentially fast (or faster) for $x \gtrsim 1$. In Fig. 5, we plot $s^\tau n(s; t_l)$ averaged over 100 different runs versus the scaling variable $sN^{-1/\tau}$ for all 21 different system sizes $N = 2^{17}, 2^{18}, \dots, 2^{36}, 2^{37}$, using $\tau = 2.06$. As is clearly seen, the collapse of the data points is *perfect* including both the power-law decay part and also the hump structure near the end. This remarkable collapse validates the single-variable FSS form of Eq. (3) without any doubt. Therefore, we now have the most precise and full information on the cluster size distribution just below the transition for large N . Comparison with the scaling collapse plot using $\tau = 2$ (see the inset of Fig. 5) leads to the definite conclusion that τ must be larger than 2.

Now we are ready to derive the upper bound of $\Delta g = g(t_u) - g(t_l)$. From $t = t_l$ to $t = t_u$, we need $\Delta L = N(t_u - t_l)$ steps (ΔL links added). One may imagine the *ideal* process to maximize the growth of the largest cluster $G(t)$, starting from the well-known cluster distribution $n(s; t_l)$ at t_l , by adding ΔL links one by one. This ideal process can be easily implemented by simply linking and merging the largest cluster with the next largest cluster at each step and repeating it till all ΔL links are exhausted. All clusters of size $s > s_\delta$ will then merge into one cluster, which becomes the largest cluster after ΔL steps. The threshold value s_δ is determined by balancing the total number of merged clusters with the total number of links added; $C \sum_{s > s_\delta} n(s; t_l) = \Delta L$ with C the total number of clusters at $t = t_l$.

During this ideal process, the largest cluster $G(t)$ grows by the amount of $C \sum_{s > s_\delta} sn(s; t_l)$. One can easily estimate $s_\delta \sim (C/\Delta L)^{1/(1-\tau)} \sim N^{\delta/(\tau-1)}$, using the single-variable FSS form of $n(s; t_l)$ of Eq. (3) with Eq. (1). Finally, we get the strict upper bound for Δg as

$$\Delta g \lesssim s_\delta^{2-\tau} \sim N^{-\delta(\tau-2)/(\tau-1)} \approx N^{-0.022}. \quad (4)$$

This shows that the order parameter jump Δg at the percolation transition should vanish as the $N \rightarrow \infty$ limit, if $\delta > 0$ and $\tau > 2$, which are undoubtedly confirmed in our numerical simulations. Therefore, we conclude that the explosive percolation transition is indeed *continuous*.

In summary, we showed that the explosive percolation transition on the complete graph is continuous by exploiting the high-precision cluster-size information at the moment of the maximum cluster heterogeneity, $t_l(N)$, approaching the asymptotic transition point t_c from below. The cluster-size distribution displays the power-law scaling with the decay exponent $\tau = 2.06(2)$, followed by a hump with a sharp cutoff $s_f \sim N^{1/\tau}$. It is explicitly shown that the existence of the single-variable finite-size scaling at $t_l(N)$ solely guarantees the continuity of the transition if $\tau > 2$. Therefore, the

scaling and the discontinuity cannot be compatible near the explosive percolation transition as in usual critical phenomena. The explosiveness originates from the huge difference in the largest-cluster-size scaling in N below and above the transition. However, it is not enough to invoke a discontinuity at the transition [22].

Our approach can be applied to many other models, including various different types of explosive percolation models to clarify the (dis-)continuity. Applications to other explosive percolation problems and also the low-dimensional cases are currently under investigation.

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- [21] The mean cluster size $\langle s \rangle = N/C = \sum_s sn(s) \approx \int ds s^{1-\tau} \sim O(1)$ for $\tau > 2$.
- [22] Discontinuous transitions were found in models with a global growth rule controlling the largest cluster only [13,14].