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LETTER TO THE EDITOR

Triviality of the critical exponents of directed self-avoiding walks on Sierpinski carpets

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Abstract. We present an analytic argument for the critical exponents ($\nu_\perp$ and $\nu_\parallel$) of the fully directed self-avoiding walks (DSAW) on a family of Sierpinski carpets. In contrast to the cases of random walks or isotropic self-avoiding walks on a fractal lattice, we find that both exponents do not depend on the fractal dimension of the underlying carpet but take a trivial value of unity. Only the correction-to-scaling exponents vary with the fractal property of the underlying lattice. Numerical simulations confirm our prediction.

The fully directed self-avoiding walks (DSAW) model has been extensively studied by many authors [1–5]. Since DSAW has a preferred direction, two independent correlation lengths $R_\perp$ and $R_\parallel$, which are perpendicular and parallel to the preferred direction respectively, can be defined. The mean square end-to-end displacements diverge algebraically with the number of steps N such that $\langle R_\perp^2 \rangle \sim N^{2\nu_\perp}$ and $\langle R_\parallel^2 \rangle \sim N^{2\nu_\parallel}$ as N becomes large. These exponents have values $\nu_\perp = \frac{1}{2}$ and $\nu_\parallel = 1$ for Euclidean lattices with spatial dimensions $d \geq 2$ [2–5]. For the isotropic walks, there is only one length scale so $\nu = \nu_\perp = \nu_\parallel$ and $\nu = \frac{1}{2}$ for ordinary random walks. For self-avoiding walks, ν varies with the spatial dimension of the underlying lattice.

When the underlying Euclidean lattice is replaced by a fractal lattice, it has been shown that the mean square end-to-end displacement for the isotropic walks scales differently from the Euclidean case (see e.g. Havlin and ben-Avraham [6] and Bouchaud and Georges [7]). Specifically the value of the exponent ν varies with the fractal dimension of the underlying lattice. The dependence of ν on the property of the underlying fractal lattice is known analytically in some special cases [6].

Based on the above observation, it is natural to ask how the scaling property of the mean square displacement of DSAW would change on a fractal lattice. Very recently, Yao and Zhuang [8] performed numerical simulations of DSAW on three Sierpinski carpets with different fractal dimensions. The generators for carpets are shown in figure 1 (here we refer to the carpets as carpet 1, 2 and 3 for (*a*), (*b*) and (*c*) respectively). On these carpets $\nu_\parallel$ is trivially 1 since DSAW exhibits a characteristic one-dimensional SAW behaviour along the preferred direction. Due to the presence of holes, the other exponent $\nu_\perp$ can have a non-trivial value, different from $\frac{1}{2}$ of the Euclidean case. Based on their numerical results, $\nu_\perp = 0.59 \pm 0.01$, 0.67 ± 0.02 and 0.83 ± 0.03 for carpets 1, 2 and 3 with fractal dimension $d_f = 1.975$, 1.892 and 1.792 respectively, Yao and Zhuang

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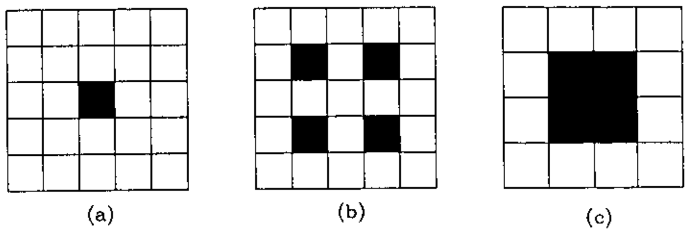

Figure 1. Carpet generators. (*a*) Carpet 1: $b=5$, $l=1$, $d_f=1.975$. (*b*) Carpet 2: $b=5$, $l=2$, $d_f=1.892$. (*c*) Carpet 3: $b=4$, $l=2$, $d_f=1.792$.

suggest that DSAW on different Sierpinski carpets belong to different universality classes. In this letter, we present an anlytic argument that the value of $\nu_\perp$ does *not* depend on the fractal dimension d_f and, in fact, $\nu_\perp$ takes a trivial value of unity. We perform numerical simulations and our numerical data strongly support the above prediction.

Consider DSAW on a family of Sierpinski carpets whose generators are specified with two indices b and l (figure 2). The fractal dimension of the carpets is given by $d_f = \ln(b^2 - l^2)/\mathrm{b}$. A walker can go either right or up by a unit length of the carpet. A trajectory of DSAW passing through the nth-generation carpet can be decomposed into (1) movements inside the $(n-1)$th-generation carpets and (2) movements along the boundary of the largest hole of the nth-generation carpet (figure 3). The largest hole in the nth-generation carpet will be called the nth-generation hole. Define A_n (B_n) as the average number of the $(n-1)$th- (nth-) generation holes encountered by a walker passing through the nth-generation carpet. Self-similarity structure of carpets guarantees the ratio of these two numbers, A_n/B_n, n-independent for large n. From now on, we will drop the subscript n for convenience. If a walker traces out the special path shown in figure 3, for example, this ratio is $b-l=4$. Along this path, a walker sees every hole of nth- and $(n-1)$th-generations located along the diagonal direction in the nth-generation carpet. For the similar paths in carpet 2 and 3, the ratio becomes again $b-1=3$, 2 respectively.

We assume that this ratio averaged over paths weighed properly by DSAW cannot be larger than the resizing factor b. Figure 4 shows 100 simulated trajectories of DSAW with the number of steps, $N=1000$, on the 4th-generation carpet 1. One can easily

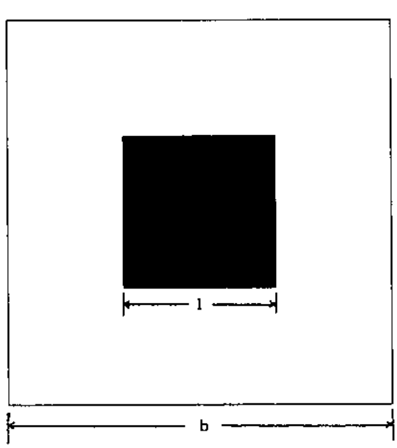

Figure 2. The generator of size b with a hole of size l.

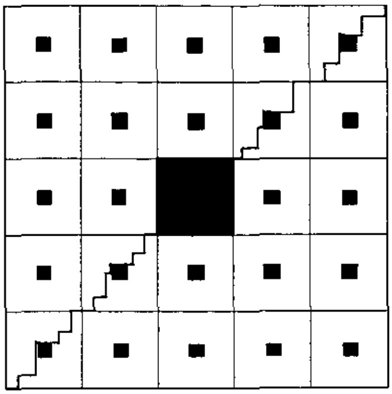

Figure 3. A special path along which a walker of DSAW can see very hole of the nth and $(n-1)$th generations located along the diagonal direction in the nth-generation carpet 1.

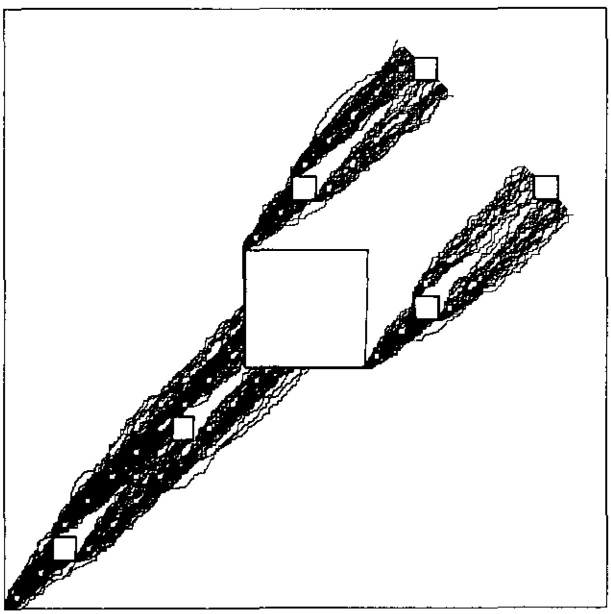

Figure 4. The trajectories of 100 random samples for 1000-step walks on the 4th-generation carpet 1. The starting position of the walkers are set at the lower left corner.

see that the fluctuation of walks perpendicular to the preferred direction is much smaller than the size of the largest hole (here, the 4th-generation hole). It implies that it is extremely unlikely for a walker to go around the largest hole without encountering it. This may be explained as follows. The Gaussian fluctuations of DSAW along the perpendicular direction is order of $\sqrt{N}$ after N steps, while the size of the holes that a walker will encounter along the preferred direction increases linearly with N. For large, N, the probability of not encountering the largest hole is exponentially small, $\sim\exp(-N)$. This suggests that the average value of the ratio A/B may be very close to $b-l$, the value for the special path in figure 3, which is consistent with our assumption of $A/B \leqslant b$.

We check this assumption numerically by studying the distribution $D(s)$ of the holes of size s, which intersect with the trajectory of DSAW. One can easily show that $D(s)$ scales as

$$D(s) \sim s^{-1-\ln(A/B)/\ln b}. \tag{1}$$

We have numerically obtained the hole size distribution to compare with (1). After starting a walker at the lower left corner on the 4th-generation carpet 1 (figure 4), we have measured the number of holes of each generation encountered by a walker during 1000 steps and averaged over 10 000 configurations. The resulting histogram is shown in figure 5. These data are fit into the power law with exponent -1.83 ± 0.04. Therefore the ratio $A/B = 3.8 \pm 0.3$, which is close to $b - l = 4$. Note that the linear fractal dimension of holes along the diagonal line of the carpet is given by $d_1 = \ln(b-l)/\ln b$. The hole size distribution along this line scales as $\sim s^{-1-d_1}$. Our numerical result suggests that the statistics of the holes encountered by a walker of DSAW are effectively the same as in the case of a one-dimensional SAW along the diagonal direction. Here, we emphasize that the actual value of A/B does not change our main result ($\nu_\perp = 1$) of this letter (but the inequality $A/B \leq b$ is crucial). This value determines the scaling behaviour of correction-to-scaling terms only.

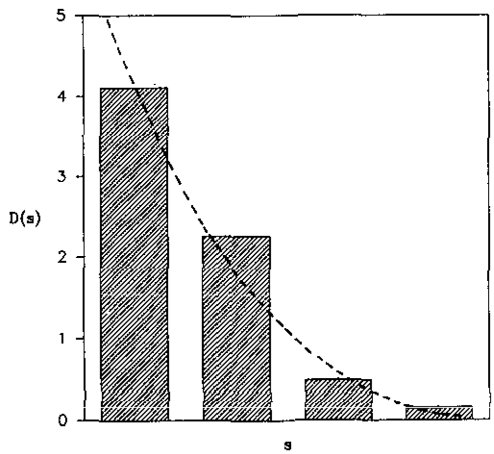

Figure 5. The histogram of distribution $D(s)$ of the holes of size s on the carpet 1. The dotted line represents the hole size distributions according to (1). $s = 1, 5, 25, 125$ from the left box.

We now proceed to estimate the mean square end-to-end displacement in the perpendicular direction. Define N_n as the number of steps needed to path through the nth-generation carpet and R_n as the fluctuation of DSAW perpendicular to the preferred direction in N_n steps. If we consider the trajectory from the lower left corner to the upper right corner, $N_n = 2b^n$. In general, a walker can start at any point along the left and bottom lines of the nth generation carpet. But N_n averaged over these starting positions is still proportional to b^n.

The estimation of R_n is a little tricky. There are two effects which contribute to R_n. One is the Gaussian fluctuation from the inherent randomness, the other is due to the presence of holes. The contribution due to the holes can be estimated as follows. As discussed before, the trajectory of DSAW passing through the nth-generation carpet can be decomposed into two parts (the $(n-1)$th-generation and the nth-generation holes). The contribution to R_n by the nth-generation holes is proportional to the size of the nth-generation holes times the number of those holes encountered by a walker, $lb^{n-1}B_n$. The average number of the $(n-1)$th-generation holes encountered by a walker with respect to that of the nth-generation holes is given by A_n/B_n. In general, this ratio for the $(n-i-1)$th- and the $(n-i)$th-generation holes is A_{n-i}/B_{n-i} for $i=0, 1, \ldots, n-1$. The contribution to R_n by a single $(n-i)$th-generation hole encountered by a walker is proportional to its size lb^{n-i-1}. As discussed before, we now assume that the ratio A_n/B_n is n-independent. If we simply add up the contributions from holes of all generations,

$$R_n \sim B_n l \sum_{i=0}^{n-1} b^{n-i-1}(A/B)^i$$

$$\sim B_n[l/(b-A/B)]b^n(1-(A/bB)^n). \qquad (2)$$

Since $A/B \leqslant b$ and B_n and $l/(b-A/B)$ are order of unity, $R_n \sim b^n$ in the large n limit. The total fluctuation, which is the sum of the hole and gaussian contribution, should be larger than b^n. On the other hand, R_n cannot be larger than the linear size of the carpet, which also scales as b^n. Due to these two constraints, R_n should also scale as b^n. Therefore the exponent $\nu_\perp$, defined as $R_n \sim N_n^{\nu_\perp}$ as $n \to \infty$, becomes a trivial value of unity.

Now consider the leading correction-to-scaling term. Substituting N_n for b^n in (2) and adding the Gaussian contribution to R_n, the perpendicular fluctuation of DSAW becomes

$$R_n \sim N_n^{\nu_\perp}(1-a_1 N_n^{-y_1}+a_g N_n^{-y_g}+\ldots) \qquad (3)$$

where $\nu_\perp = 1$ and a_1, a_g are constants. The subdominant exponent of the hole contributions $y_1 = -1+\ln(A/B)/\ln b$ and the leading exponent of the Gaussian contributions $y_g = \frac{1}{2}$. If $A/B = b-1$, then $y_1 = -1+d_1$ where d_1 is the one-dimensional fractal dimension discussed previously. Note that, for carpets 1, 2 and 3, $d_1 = 0.861$, 0.683 and 0.5 respectively. Therefore the leading correction-to-scaling term in R_n comes from the subdominant term of the hole contributions in all three carpets considered here. This correction term decreases (y_1 increases) as d_1 becomes smaller. This may explain why the numerical value of $\nu_\perp$ obtained by Yao and Zhuang [8] monotonically increases to 1, the true value, as d_1 becomes smaller.

We have performed numerical simulations to confirm our analytic arguments. First, we have generated the Sierpinski carpets using the generators as shown in figure 1. In order to investigate the properties of DSAW on a given carpet, we randomly choose a point and start a two-choice random walker. A walker can go right or up by a unit length with equal probability. If a walker touches the boundary of the carpet, we discard the walker and start a new walker at a newly chosen position. We have continued these procedures until getting 10 000 configurations for each given steps N up to 256 for the carpets 1 and 2, but up to 128 for carpet 3 with relatively small lattice size. The end-to-end displacements, $R_\perp$, are measured for each configuration in the direction perpendicular to the preferred direction. The squares of the displacements are averaged over 10 000 configurations. In figure 6, we plot $\log_2\langle R_\perp^2\rangle$ versus $\log_2 N$ for all three carpets. The lines connecting successive data points are quite straight and their slopes are calculated by using least-squares fitting. The results are $\nu_\perp = 1.04 \pm 0.04$, 1.02 ± 0.03 and 1.02 ± 0.03 for the carpets 1, 2 and 3, respectively. Our numerical results clearly

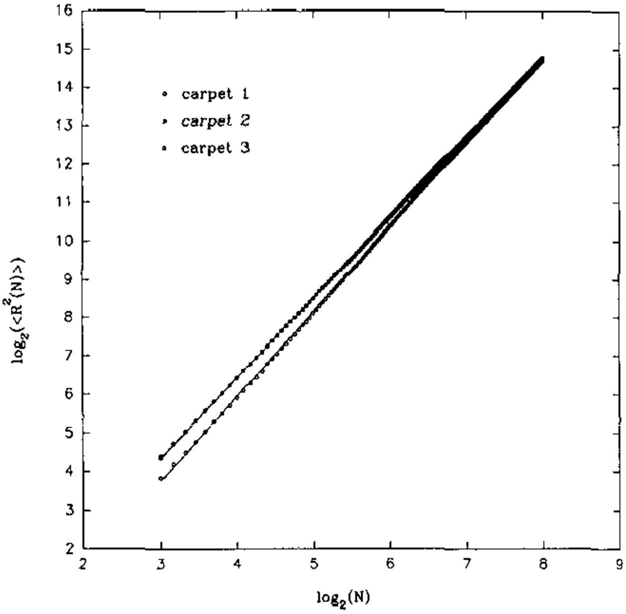

Figure 6. Plots of $\log_2\langle R_\perp^2(N)\rangle$ versus $\log_2 N$ for carpet 1, 2 and 3. The data for carpet 2 and 3 almost coincide with each other. The extrapolated values of the half slopes are 1.04 ± 0.04, 1.02 ± 0.03 and 1.02 ± 0.03 for each carpet.

confirm our prediction of $\nu_\perp = 1$ for all carpets. However, at present, we could not extract the reasonable information about the correction-to-scaling terms due to statistical errors. In order to confirm our prediction about the leading subdominant exponent, one needs to perform much more extensive simulations with large n. Another note worthy of mention is about taking periodic boundary conditions. We attempted to apply periodic boundary conditions to get longer steps of walks. But, as we expected, this scheme yields only Euclidean crossover to give the value of $\nu_\perp$ close to $\frac{1}{2}$.

In conclusion, we argue that $\nu_\perp$ and $\nu_\parallel$ of DSAW on Sierpinski carpets should be the trivial value of unity, independent of the fractal dimensions of the underlying carpets. In fact they depend only on the one-dimensional characteristic of the carpet along the preferred direction of DSAW. We summarize our argument for $\nu_\perp = 1$ as follows. There are two contributions to the fluctuation along the perpendicular direction ($R_\perp$). One is the Gaussian randomness, the other is the presence of the holes. The Gaussian fluctuation perpendicular to the diagonal direction is of the order of $\sqrt{N}$, where N is the total number of steps. But the size of the largest hole increases linearly with N along the diagonal direction (the preferred direction of DSAW). Therefore the random walker eventually has to encounter the largest hole, increasing the fluctuations of $R_\perp$ by the amount proportional to the linear size of that hole (see figure 4). Therefore $R_\perp$ must scale as the linear size of the holes along the preferred direction, hence $\nu_\perp = 1$.

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