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The statistical mechanics of the coagulation–diffusion process with a stochastic reset

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Abstract

The effects of a stochastic reset, to its initial configuration, is studied in the exactly solvable one-dimensional coagulation–diffusion process. A finite resetting rate leads to a modified non-equilibrium stationary state. If, in addition, the input of particles at a fixed given rate is admitted, then a competition between the resetting and the input rates leads to a non-trivial behaviour of the particle-density in the stationary state. From the exact inter-particle probability distribution, a simple physical picture emerges: the reset mainly changes the behaviour at larger distance scales, while at smaller length scales, the non-trivial correlation of the model without a reset dominate.

Keywords: stochastic processes, exactly solvable model, population dynamics
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(Some figures may appear in colour only in the online journal)

1. Introduction

Stochastic resets occur quite commonly in very distinct situations. For example, consider a network of tidal channels on a beach. From time to time, the network is washed out by a larger wave. What would be the average properties of such a network? How do they differ close to the water line (when resetting due to waves is frequent)? And, how do they differ farther inland? Another often-met instance concerns the search for an object. After having searched in vain for some time, a frequently-used search strategy consists of returning to the beginning and starting afresh, which continues until the object is found. In their remarkable work, Evans and Majumdar (EM) [16] explored the consequences of stochastic resetting in the simple diffusion of a single random walker. They considered a random walk along a line, with a time-dependent position $x(t)$ and starting from an initial position $x(0) = x_0 \neq 0$, and also with an absorbing trap at the origin $x = 0$. While undergoing the random walk, the particle is reset to its initial position with a rate r. EM showed that the statistical properties of the random walk are drastically altered by the resetting. For example, in the long-time limit, the stationary distribution of the particle with reset is no longer Gaussian and the mean time to find a target at the origin becomes finite whenever $r > 0$ and actually has a minimum at some non-trivial value $r^* \neq 0$ [16]. Various aspects of searches with reset have been analysed recently [2, 5, 9, 28, 30, 35] and these considerations have also been extended to the consideration of teams of independent researchers [17–19, 29].

Here, we are interested in analysing a simple model of *interacting* particles, subject to a stochastic reset to its initial configuration. We shall choose here the *coagulation–diffusion process*, which in one spatial dimension is exactly solvable through the method of empty intervals [6] and whose properties have been analysed profoundly in the past; for example, [3, 7, 8, 10–15, 20, 23, 25, 27, 31, 32, 34, 37, 38]. The coagulation-diffusion process can be defined in terms of particles that move diffusively on an infinite chain, such that each site is either empty or occupied by a single particle. If a particle makes an attempt to jump to a site that is already occupied, it disappears from the system with probability one, according to $A+A \to A$. It is well-known that this system can be exactly solved through the method of empty intervals, where the central quantity is the probability $E_n(t)$ that n consecutive sites are empty at time t. The time-dependent average particle-density is then given by $\rho(t) = (1 - E_1(t))/a$, whereas the E_n satisfy for all $n \geqslant 1$ the equation [6, 7, 10, 12, 22][3]

$$\partial_t E_n(t) = \frac{2D}{a^2}\left(E_{n-1}(t) + E_{n+1}(t) - 2E_n(t)\right), \quad E_0(t) = 1, \quad E_\infty(t) = 0 \tag{1.1}$$

where a is the lattice constant and D the diffusion constant. In the continuum limit, one has $E_n(t) \to E(t,x)$, the particle-density becomes $\rho(t) = -\,\partial_x E(t,x)|_{x=0}$ and, finally, (1.1) turns into $\left(\partial_t - 2D\partial_x^2\right) E(t,x) = 0$ with the boundary conditions $E(t,0) = 1$, $E(t,\infty) = 0$. How to treat these boundary conditions directly has only recently been understood [14]. The resulting long-time behaviour $\rho(t) \sim t^{-1/2}$ has been confirmed in several experiments involving excitons moving on polymer chains [24, 26] or carbon nanotubes [4, 33, 36]. The strong mathematical similarity of these equations with the equations for the probability distribution of a random walker [16] initially motivated us to consider a stochastic reset in the coagulation–diffusion process.

We now define the *coagulation–diffusion process with a stochastic reset* (CDPR): consider a chain with $\mathcal{N}$ sites, each of which can be occupied by at most one particle. The particles perform random hoppings to nearest-neighbour sites such that upon encounter of two particles, the arriving particle disappears. The stochastic reset is described by a given set of probabilities F_n for having n consecutive empty sites[4]. A sweep of the lattice consists of $\mathcal{N}$ steps of the microscopic dynamics. In each step, one first chooses a particle. This particle either diffuses with probability $\mathcal{P}_g = \frac{D}{2D+r/\mathcal{N}}$ to the left, or else with probability $\mathcal{P}_d = \mathcal{P}_g$ to the right or,

[3] A change in $E_n(t) = \mathbf{P}(\boxed{\text{n}};t)$ can only arise if at the boundary of the empty interval, a particle either jumps into the interval or else moves one step further to the side to increase the size of the interval. Hence, $\partial_t E_n = 2D\left(\mathbf{P}(\bullet\boxed{\text{n-1}};t) - \mathbf{P}(\bullet\boxed{\text{n}};t)\right)$. The factor 2 arises since analogous processes can occur at both ends of the interval. Finally, one has $\mathbf{P}(\bullet\boxed{\text{n}};t) + \mathbf{P}(\circ\boxed{\text{n}};t) = \mathbf{P}(\boxed{\text{n}};t)$ and $\mathbf{P}(\circ\boxed{\text{n}};t) = \mathbf{P}(\boxed{\text{n+1}};t)$ and (1.1) follows.

[4] For example, for a configuration of uncorrelated particles such that each single site is occupied with probability p, one has $F_n = (1-p)^n$. Consistency implies that $F_0 = 1$ always.

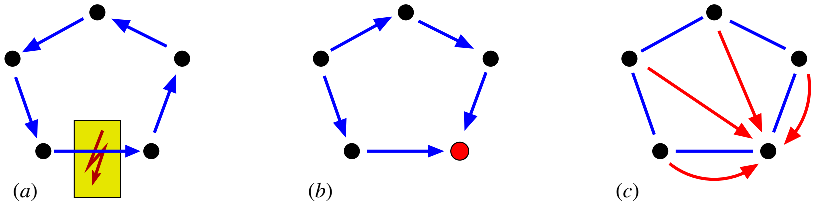

Figure 1. Schematic illustration of different kinds of non-equilibrium stationary states. The probability currents are indicated by the arrows. (*a*) Closed loop of probability currents, driven through the coupling to external engines. (*b*) Absorbing stationary state (red dot). (*c*) Network of probability currents, modified through a reset to a certain configuration, the additional probability currents are indicated by red arrows.

finally, the entire system is reset to the configuration F_n with probability $\mathcal{P}_r = \frac{r/\mathcal{N}}{2D+r/\mathcal{N}}$. In terms of the empty-interval probabilities $E_n(t)$, the equation of motion (1.1) is modified as follows by the stochastic reset, for $n \geqslant 1$

$$\partial_t E_n(t) = \frac{2D}{a^2}\left(E_{n-1}(t) + E_{n+1}(t) - 2E_n(t)\right) - rE_n(t) + rF_n, \quad E_0(t) = 1, \quad E_\infty(t) = 0 \tag{1.2}$$

which generalizes the problem of the stochastic resetting of a single random walker as formulated by EM [16][5]. Besides being a case study of the influences of a stochastic reset, the results of this study might also be considered from a different point of view, such as by studying how a non-equilibrium system may be set up. A very common way to do this to allow for non-vanishing *probability currents* between the states of the system (which can be physically realized through coupling to external reservoirs), leading to closed loops between states, see figure 1(*a*). For a detailed review of the properties of the non-equilibrium stationary states, see [39] and the references therein. An alternative possibility occurs for *absorbing stationary states*, where the system's evolution goes towards a single absorbing state it cannot leave any more (figure 1(*b*)); see, for example, [21] and references therein for an overview. Here, and following EM, we consider what might be happening if the transition probabilities between states are changed such that the system can return with a certain probability r to its initial state, see figure 1(*c*).

This paper is organized as follows. The exact solution of equation (1.2) is derived in section 2. On a discrete chain, a detailed comparison with direct numerical simulation of the CDPR establishes that (1.2) is indeed the correct analytical description of the coagulation–diffusion process with reset. As seen before by EM for the random walk with reset, the stationary density of single particles and pairs is non-vanishing whenever $r > 0$ while the entire probability distribution is modified, which can be illustrated through the explicit expressions for the $E_n(t)$. As a preparation for later generalization, the continuum limit of the same model is derived and leads to the same qualitative conclusions. In section 3 we extend the model by admitting in addition the possibility of particle-input on the lattice, at a fixed rate λ. It turns out that the input and reset interact in a rather non-intuitive way, which leads to a complex and non-monotonous dependence of the stationary particle-density on these parameters. This surprising result will be further illuminated in section 4. Indeed, the shape of the distribution of the size of the empty intervals between particles reproduces the non-trivial correlations

[5] In the context of the experiments on the exciton kinetics [4, 24, 26, 33, 36], one might consider an extension where the chain is re-populated by a laser beam at some regular rate r, rather than letting the particles disappear before the system is refilled.

of coagulation–diffusion (with or without input) for small intervals, whereas the distribution of larger empty intervals is governed by the choice of the reset. Conclusions are given in section 5. An appendix discusses details of the choice of the transition rates in Monte Carlo simulations.

2. Model

The 1D coagulation–diffusion process with a reset (CDPR), as defined in the introduction, can be treated analytically by introducing the empty-interval probability $E_n(t)$, see [6] and the references therein. The equation of motion is given by (1.2). In this section, we shall first compute the $E_n(t)$ exactly on an infinite chain and derive from this the particle- and pair-densities. A detailed comparison with a direct numerical simulation of the CDPR will illustrate that (1.2) gives the correct analytical description. We shall also study the continuum limit of the model.

A useful alternative route towards the *stationary* solution of (1.2) starts from the time-dependent equation (1.1), without a reset. Then, the Laplace transform $\overline{E}_n(r) := \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-rt} E_n(t)$, along with the initial condition $E_n(0) = rF_n$ and with the formal replacement $t \mapsto r$ obeys the stationary equation (1.2) with a vanishing left-hand side, and the boundary condition $\overline{E}_0(r) = r^{-1}$. Although we shall not follow this route here, this idea might become useful to study the effects of a reset in more general situations.

2.1. Discrete case

The solution to (1.2), together with the non-trivial boundary condition $E_0(t) = 1$, can be derived by admitting an analytical continuation to negative indices via $E_{-n}(t) = 2 - E_n(t)$ [14]. Similarly, we shall define $F_{-n} := 2 - F_n$ for the resetting distribution. Then, the generating function

$$G(z,t) := \sum_{m=-\infty}^{\infty} z^m E_m(t) \tag{2.1}$$

obeys, because of (1.2), the equation

$$\partial_t G(z,t) = \frac{2D}{a^2}\left(z + \frac{1}{z} - \left(2 + \frac{r}{2D}\right)\right) G(z,t) + rF(z) \tag{2.2}$$

where $F(z) = \sum_{m\in\mathbb{Z}} z^m F_m$ is the generating function of the resetting distribution F_n. Equation (2.2) is almost automatically solved. Setting the lattice constant $a = 1$ from now on, the full solution can be decomposed as $E_n(t) = E_n^{(1)}(t) + E_n^{(2)}(t)$, with the 'homogeneous' solution (I_n is a modified Bessel function [1])

$$E_n^{(1)}(t) = \mathrm{e}^{-(4D+r)t} \sum_{m=-\infty}^{\infty} E_m(0) I_{n-m}(4Dt) \tag{2.3}$$

and the 'inhomogeneous' part

$$E_n^{(2)}(t) = r\int_0^t \mathrm{d}t'\, \mathrm{e}^{-(4D+r)t'} \sum_{m=-\infty}^{\infty} F_m I_{n-m}(4Dt'). \tag{2.4}$$

Using the analytical continuations $E_{-n}(0) = 2 - E_n(0)$ and $F_{-n} = 2 - F_n$, we obtain

$$E_n^{(1)}(t) = \mathrm{e}^{-(4D+r)t} \left[\sum_{m=1}^{\infty} E_m(0)\,(I_{n-m}(4Dt) - I_{n+m}(4Dt)) + 2\sum_{m=1}^{\infty} I_{n+m}(4Dt) + I_n(4Dt) \right]$$

$$E_n^{(2)}(t) = r\int_0^t \mathrm{d}t'\, \mathrm{e}^{-(4D+r)t'} \left[\sum_{m=1}^{+\infty} F_m (I_{n-m}(4Dt') - I_{n+m}(4Dt')) + 2\sum_{m=1}^{\infty} I_{n+m}(4Dt') + I_n(4Dt') \right]. \tag{2.5}$$

It is straightforward to check that these $E_n(t) = E_n^{(1)}(t) + E_n^{(2)}(t)$ indeed solve the equations of motion and obey the required boundary condition. We also observe that only the F_n with $n \geqslant 1$ enter into the final solution. Since $E_n^{(1)}(t)$ simply reproduces the well-known solution without a reset [14, 37], clearly $E_n^{(2)}(t)$ gives a contribution to the reset. This illustrates how the stochastic reset modifies the entire probability-distribution of the states in the CDPR.

With a knowledge of these empty-interval probabilities, one can derive the particle-density $\rho(t) = P(\bullet)$ and the pair-density $p(t) = P(\bullet\bullet)$ using the following relations

$$\rho(t) = 1 - E_1(t)$$
$$p(t) = 1 - 2E_1(t) + E_2(t).$$

The particle-density $\rho(t)$ reads

$$\rho(t) = \mathrm{e}^{-(4D+r)t} \left(I_0(4Dt) + I_1(4Dt) - \sum_{m=1}^{\infty} \frac{mE_m(0)I_m(4Dt)}{2Dt} \right) + r\int_0^t \mathrm{d}t'\, \mathrm{e}^{-(4D+r)t'} \left(I_0(4Dt') + I_1(4Dt') - \sum_{m=1}^{\infty} \frac{mF_m I_m(4Dt')}{2Dt'} \right) \tag{2.6}$$

and the pair-density $p(t)$ is given by

$$p(t) = \mathrm{e}^{-(4D+r)t} \left[I_0(4Dt) - I_2(4Dt) - 2\sum_{m=1}^{\infty} \frac{mE_m(0)I_m(4Dt)}{2Dt} + \sum_{m=1}^{\infty} E_m(0) \left(\frac{(m-1)I_{m-1}(4Dt)}{2Dt} + \frac{(m+1)I_{m+1}(4Dt)}{2Dt} \right) \right] + r\int_0^t \mathrm{d}t'\, \mathrm{e}^{-(4D+r)t'} \left[I_0(4Dt') - I_2(4Dt') - 2\sum_{m=1}^{\infty} \frac{mF_m I_m(4Dt')}{2Dt'} + \sum_{m=1}^{\infty} F_m \left(\frac{(m-1)I_{m-1}(4Dt')}{2Dt'} + \frac{(m+1)I_{m+1}(4Dt')}{2Dt'} \right) \right]. \tag{2.7}$$

Using several relations from [1], the stationary concentration is found from the second term of equation (2.6)

$$\rho_{\text{stat,disc}} = \frac{\sqrt{r(r+8D)} - r}{4D} - \frac{r}{2D} \sum_{m=1}^{\infty} \left(\frac{1-p}{2} \right)^m \left(\frac{4D}{r+4D} \right)^m \times {}_2F_1 \left(\frac{m}{2}, \frac{m+1}{2}; m+1; \left(\frac{4D}{r+4D} \right)^2 \right). \tag{2.8}$$

In figure 2, the analytic results (2.6), (2.7) are compared with simulational results obtained from the CDPR, as defined in section 1. Here, a reset to a random uncorrelated configuration of particles of mean concentration p with an empty-interval distribution $F_n = (1-p)^n$ was

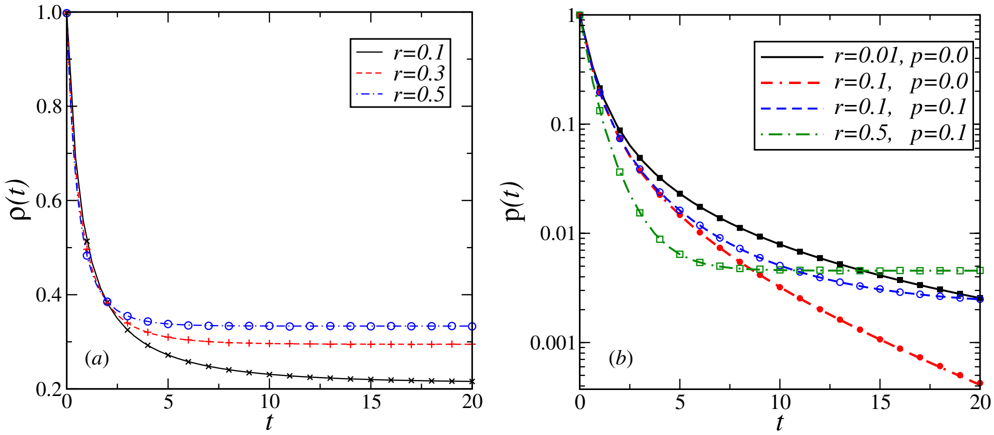

Figure 2. Left panel (a): particle-density $\rho(t)$ over against time t for different values of the reset parameter r and particle concentration $p = 0.5$. Right panel (b): pair-density $p(t)$ over against time for various values of the reset parameter r and the particle concentration p. Here, an uncorrelated reset configuration $F_n = (1-p)^n$ was used. In both panels, the full line represents the analytic solution while the symbols show the Monte Carlo simulations.

used. We find a clear agreement that confirms the correctness of the equation of motion (1.2) and permits identification with the lattice model.

In the left panel of figure 2, the relaxation of the particle-density $\rho(t)$ towards its non-vanishing stationary value (since $p \neq 0$ and $r \neq 0$) is shown. One also sees that the relaxation towards a stationary value is exponentially fast, instead of following a slow algebraic decay $\rho(t) \sim t^{-1/2}$ obtained without reset. In the right panel of figure 2, an analogous behaviour is found for the pair-density $p(t)$ when the reset is made to a non-vanishing concentration, $p = 0.1$. On the other hand, if one considers a reset to an empty lattice ($p = 0$), one clearly sees that the relaxation has become exponential instead of following a slow decay $p(t) \sim t^{-1}$, which would hold true in the absence of a reset.

2.2. Continuum limit

In the continuum limit, the equation of motion (1.2) becomes

$$\partial_t E(t,x) = 2D\partial_x^2 E(t,x) - rE(t,x) + rF(x) \tag{2.9}$$

with the boundary conditions $E(t,0) = 1$ and $E(t,\infty) = 0$. One way to solve this equation is to separate the empty-interval probability as $E(t,x) = \frac{1}{2}f(x) + b(t,x)$ such that $f(x)$ will give the stationary solution and $b(t,x)$ will describe the relaxation towards it. Then, the equation for the stationary probability is

$$f''(x) - \alpha^2 f(x) + 2\alpha^2 F(x) = 0; \quad f(0) = 2, \quad f(\infty) = 0 \tag{2.10}$$

where

$$\alpha^2 := \frac{r}{2D}. \tag{2.11}$$

The general solution of (2.10) is readily found by a variation of constants. Taking the two boundary conditions into account, a straightforward calculation gives

$$f(x) = 2\,\mathrm{e}^{-\alpha x} + \alpha \int_x^\infty \mathrm{d}x'\, F(x')\,\mathrm{e}^{\alpha(x-x')} + \alpha \int_0^x \mathrm{d}x'\, F(x')\,\mathrm{e}^{\alpha(x'-x)} - \alpha \int_0^\infty \mathrm{d}x'\, F(x')\,\mathrm{e}^{-\alpha(x+x')}. \tag{2.12}$$

To go further, a resetting distribution $F(x)$ must be specified. In the case of a random distribution where the particles have a probability p to be on a site, the empty-interval probability is given by $E_n(t) = (1-p)^n$ and, in the continuum limit $x = na$ and $p \to 0$, the resetting distribution reads $F(x) = \mathrm{e}^{-cx}$, with the reset density $c = -\ln(1-p) \simeq p + \mathrm{O}(p^2)$. Then, the stationary part $f(x)$ is

$$f(x) = \left(2 - \frac{\alpha}{\alpha+c}\right)\mathrm{e}^{-\alpha x} + \frac{\alpha}{\alpha+c}\,\mathrm{e}^{-cx} + \frac{\alpha}{\alpha-c}\left(\mathrm{e}^{-(c-\alpha)x} - 1\right)\mathrm{e}^{-\alpha x} \tag{2.13}$$

such that the stationary concentration ρ_{stat} is given by

$$\rho_{\text{stat}} = -\frac{1}{2}\frac{\partial f(x)}{\partial x}\bigg|_{x=0} = \frac{\alpha c}{\alpha+c}. \tag{2.14}$$

It can be checked that, in the limit $c \ll 1$ of small concentration, this expression is consistent with the discrete solution (2.8).

The dynamical part $b(t,x)$ of (2.9) satisfies the following equation

$$\partial_t b(t,x) = 2D\partial_x^2 b(t,x) - rb(t,x) \tag{2.15}$$

with the boundary conditions $b(t,0) = b(t,\infty) = 0$. Using a sine Fourier transform, one can easily find the solution

$$b(t,x) = \sqrt{\frac{\pi}{2Dt}}\,\mathrm{e}^{-2D\alpha^2 t}\int_0^\infty \mathrm{d}x'\; b_0(x')\left[\mathrm{e}^{-\frac{(x-x')^2}{8Dt}} - \mathrm{e}^{-\frac{(x+x')^2}{8Dt}}\right] \tag{2.16}$$

where $b_0(x)$ is the initial condition, which can also be decomposed as $b_0(x) = E_0(x) - \frac{1}{2}f(x)$ where $f(x)$ will give the universal term and $E_0(x)$ will give the initial-condition-dependent term. Replacing $f(x)$ by its expression in (2.13), the universal term of the concentration reads

$$\begin{aligned}\rho_{\text{uni}}(t) &= -\frac{\partial b_{\text{uni}}(x)}{\partial x}\bigg|_{x=0}\\ &= \sqrt{\frac{2\pi}{Dt}}\,\mathrm{e}^{-2D\alpha^2 t} + \frac{2\pi\alpha c}{c^2-\alpha^2}\,\mathrm{e}^{-2D\alpha^2 t}\left(\alpha\,\mathrm{e}^{2c^2Dt}\mathrm{erfc}(c\sqrt{2Dt}) - c\,\mathrm{e}^{2D\alpha^2 t}\,\mathrm{erfc}(\alpha\sqrt{2Dt})\right).\end{aligned} \tag{2.17}$$

The initial-condition-dependent term becomes in the special case of initially uncorrelated particles, when $E_0(x) = \mathrm{e}^{-cx}$,

$$\rho_{\text{dep}}(t) = 2\pi c\,\mathrm{e}^{-2D(\alpha^2-c^2)t}\,\mathrm{erfc}(c\sqrt{2Dt}) - \sqrt{\frac{2\pi}{Dt}}\,\mathrm{e}^{-2D\alpha^2 t}. \tag{2.18}$$

Hence, the full particle-density becomes

$$\rho(t) = \rho_{\text{stat}} + \rho_{\text{uni}}(t) + \rho_{\text{dep}}(t) \overset{t\to\infty}{\simeq} \frac{\alpha c}{\alpha+c} + \mathrm{O}(t^{-1/2}\exp(-2D\alpha^2 t)). \tag{2.19}$$

The introduction of the reset has led to a non-vanishing stationary particle-density ρ_{stat}. For a fixed value of the initial concentration c, ρ_{stat} increases monotonously as a function of the reset rate α. This is qualitatively analogous to the behaviour of a single random walk with reset [16]. Furthermore, the leading approach towards this new non-equilibrium stationary state is for $\alpha > 0$ exponentially fast, but reverts in the limit $\alpha \to 0$ to the standard slow relaxation $\mathrm{O}(1/\sqrt{t})$ for the coagulation–diffusion model without reset.

3. Stochastic reset and input of particles

We now extend the model and allow the deposition ('input') of particles on the lattice, with a fixed rate λ. Without the reset, this has been treated long ago [6, 10]. In view of the technical complexities, we shall only treat the case of the continuum limit, where the equation of motion of the empty intervals becomes, for $x \geqslant 0$,

$$\partial_t E(t,x) = 2D\partial_x^2 E(t,x) - \lambda x E(t,x) - rE(t,x) + rF(x) \tag{3.1}$$

and is subject to the boundary and initial conditions

$$E(t,0) = 1, \quad E(t,\infty) = 0, \quad E(0,x) = E_0(x). \tag{3.2}$$

Again, one may separate this as $E(t,x) = \frac{1}{2}f(x) + b(t,x)$ such that $f(x)$ will give the stationary solution and $b(t,x)$ will describe the relaxation towards it. Introducing the abbreviations

$$\alpha^2 := \frac{r}{2D}, \quad \beta^3 := \frac{\lambda}{2D}, \quad \mu := \frac{\alpha^2}{\beta^3} = \frac{r}{\lambda} \tag{3.3}$$

the equation for the stationary empty-interval distribution becomes

$$f''(x) - \beta^3(x+\mu)f(x) + 2\alpha^2 F(x) = 0; \quad f(0) = 2, \quad f(\infty) = 0. \tag{3.4}$$

This may be solved by the standard variation of constants, although the expressions become quite lengthy. The general solution of the homogeneous part of (3.4) is

$$f_{\text{hom}}(x) = c_1\frac{\sqrt{3}}{2}(\text{Bi}(\beta(x+\mu)) - \sqrt{3}\text{Ai}(\beta(x+\mu))) + c_2\pi\sqrt{3}\text{Ai}(\beta(x+\mu)) \tag{3.5}$$

where Ai and Bi are Airy functions [1] and $c_{1,2}$ are constants. Then, the general solution of (3.4) can be written in the form

$$\begin{aligned} f(x) = c_1\frac{\sqrt{3}}{2}&(\text{Bi}(\beta(x+\mu)) - \sqrt{3}\text{Ai}(\beta(x+\mu))) + c_2\pi\sqrt{3}\text{Ai}(\beta(x+\mu)) \\ &+ \frac{2\alpha^2}{\beta}[\text{Bi}(\beta(x+\mu)) - \sqrt{3}\text{Ai}(\beta(x+\mu))]\int_x^\infty \mathrm{d}x'\, F(x')\text{Ai}(\beta(x'+\mu)) \\ &+ \frac{2\alpha^2}{\beta}\text{Ai}(\beta(x+\mu))\int_0^x \mathrm{d}x'\, F(x')[\text{Bi}(\beta(x'+\mu)) - \sqrt{3}\text{Ai}(\beta(x'+\mu))]. \end{aligned}$$

Using the asymptotic behaviour of the Airy functions [1], it is easy to see that $f(\infty) = 0$ implies that $c_1 = 0$ and the second boundary condition $f(0) = 2$ fixes c_2. This leads to

$$\begin{aligned} f(x) = \frac{2\text{Ai}(\beta(x+\mu))}{\text{Ai}(\beta\mu)} &- \frac{2\pi\alpha^2}{\beta}\frac{\text{Bi}(\beta\mu)}{\text{Ai}(\beta\mu)}\text{Ai}(\beta(x+\mu))\int_0^\infty \mathrm{d}x'\, F(x')\text{Ai}(\beta(x'+\mu)) \\ &+ \frac{2\pi\alpha^2}{\beta}\text{Bi}(\beta(x+\mu))\int_x^\infty \mathrm{d}x'\, F(x')\text{Ai}(\beta(x'+\mu)) \\ &+ \frac{2\pi\alpha^2}{\beta}\text{Ai}(\beta(x+\mu))\int_0^x \mathrm{d}x'\, F(x')\text{Bi}(\beta(x'+\mu)). \end{aligned} \tag{3.6}$$

3.1. The stationary density of particles

The stationary density of particles is obtained from the previous equation (3.6), which can be written down in a scaling form that also involves the average particle-density $c = -F'(0)$ in the reset configuration $F(x)$, and reads

$$\rho_{\text{stat}} = -\frac{1}{2}\left.\frac{\partial f(x)}{\partial x}\right|_{x=0} = cP\left(\frac{c}{\beta}, \beta\mu\right) \tag{3.7}$$

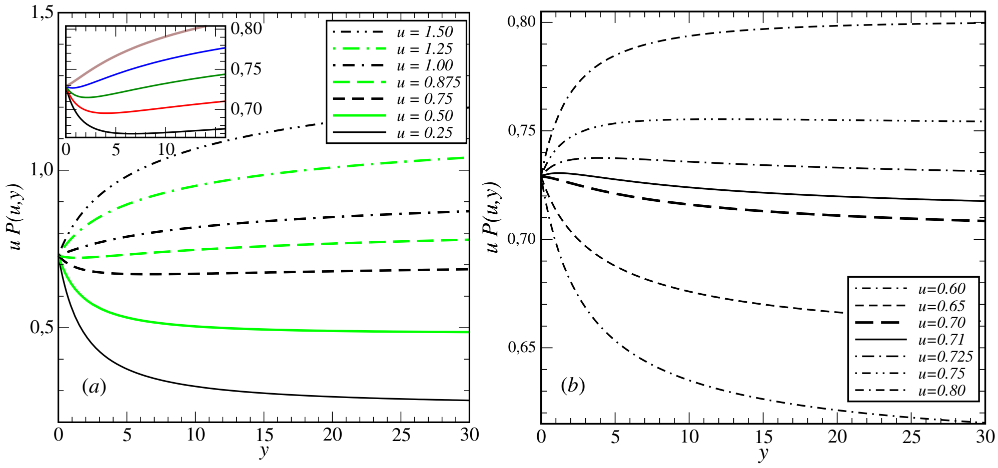

Figure 3. Plot of the scaling function $uP(u, y)$ as a function of y and for different values of u. The left panel (a) uses the reset function $F(x) = \exp(-cx)$, appropriate for uncorrelated particles; the inset shows the same plot for the values $u = [0.75, 0.80, 0.85, 0.90, 0.95]$ from bottom to top. In the right panel (b), the scaling function $uP(u, y)$ for the choice $F(x) = \mathrm{erfc}(\frac{1}{2}\sqrt{\pi}\, cx)$ is shown for comparison.

Table 1. The limit behaviour of the scaling function $uP(u, y)$ in equation (3.8), for small and large values of u and y, respectively.

$(2\pi)^{-1}3^{5/6}\Gamma(2/3)^2 + 3^{2/3}\Gamma(2/3)(4\pi)^{-1}(3\Gamma(2/3)^3 - 4\pi^2/3)y$	$u \to 0$	$y \to 0$
$y^{-1} - u/cF'(0)$	$u \to 0$	$y \to \infty$
$(2\pi)^{-1}3^{5/6}\Gamma(2/3)^2 + (2\pi)^{-2}3^{5/3}\Gamma(2/3)^3 y$	$u \to \infty$	$y \to 0$
$y^{1/2}$	$u \to \infty$	$y \to \infty$

with the explicit scaling function

$$P(u, y) := -\frac{1}{u}\frac{\mathrm{Ai}'(y)}{\mathrm{Ai}(y)} - \pi y\left(\mathrm{Bi}'(y) - \mathrm{Ai}'(y)\frac{\mathrm{Bi}(y)}{\mathrm{Ai}(y)}\right)\int_0^\infty \mathrm{d}Y\, F(uY/c)\mathrm{Ai}(Y + y). \tag{3.8}$$

Herein, the first scaling variable $u = c/\beta$ measures the ratio of the particle-density of the reset configuration with respect to the stationary density without reset and the second scaling variable $y = \beta\mu = (\alpha/\beta)^2$ is a function of the ratio of the reset rate with the input rate. The scaling function $P = \rho_{\mathrm{stat}}/c$ itself directly measures the stationary density in units of the reset density c. The asymptotic behaviour of the scaling functions for u and y small or large are listed in table 1 (remarkably, the limits are independent of the choice for the reset distribution $F(x)$). For $y \to 0$, one always recovers the known stationary particle-density of the case without reset [6, 10], as expected. However, the qualitative behaviour of $P(u, y)$ as a function of y changes according to the fixed value of u. When $u \ll 1$, $P(u, y)$ will monotonously decrease as a function of y, whereas, for $u \gg 1$, one observes a monotonous increase. From table 1, this can be read off analytically from the small-y behaviour

$$uP(u, y) \simeq \begin{cases} 0.729 - 0.531y; & \text{if } u \to 0 \\ 0.729 + 0.392y; & \text{if } u \to \infty. \end{cases} \tag{3.9}$$

The surprisingly complex behaviour of $P(u, y)$ is further illustrated in figure 3 and also depends in a subtle way on the choice of the resetting configuration $F(x)$. We begin with the

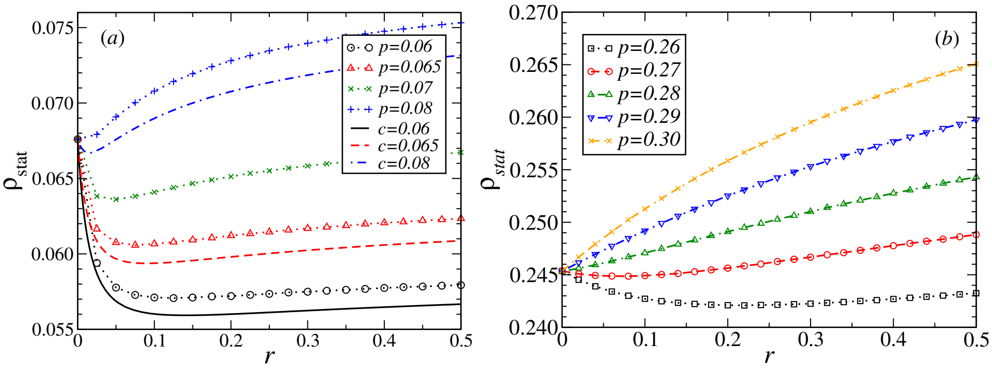

Figure 4. Plot of the stationary probability $\rho_{\rm stat}$ as a function of the reset parameter r towards uncorrelated particles for two values of the input parameter: (a) left panel $\lambda = 0.0008$, (b) right panel $\lambda = 0.04$. In the left panel, the full lines without symbols give the analytical solution (3.7) in the continuum limit.

case $F(x) = e^{-cx}$ of uncorrelated particles with concentration c. From figure 3(a), it can be seen that there is also an intermediate range $u \approx 0.8$–0.9, when $P(u, y)$ is a non-monotonous function of y. From the inset in figure 3(a), it can be seen that $P(u, y)$ goes through a minimum before a final growing regime for y that is sufficiently large is reached. A local analysis shows that $\partial P(u, y)/\partial y < 0$ for $u = u_c :=\leqslant 0.929\,5765\ldots$, which means that this minimum exists for all $u < u_c$.

For different choices of $F(x)$, one may encounter different scenarios. In figure 3(b), we show the results for the choice $F(x) = \mathrm{erfc}(\frac{1}{2}\sqrt{\pi}\, cx)$.[6] With neither a reset nor an input, the system will converge towards this distribution, with a certain (time-dependent) particle-density c. While for $u \ll 1$ and for $u \gg 1$, the qualitative behaviour is analogous to the one seen before, a non-monotonous behaviour now occurs in the middle region $u \approx 0.7 - 0.75$, with $u_c = 0.701\,0036\ldots$. However, for u not too far above u_c, the scaling function now shows a maximum.

The analytical results of this section were obtained in the continuum limit. The results from a direct numerical simulation of the CDPR with input are shown in figure 4, for two values of the input rate λ and in the case of a reset towards uncorrelated particles. Qualitatively, the behaviour of the stationary density is analogous to the one seen in figure 3(a) and, hence, is in qualitative agreement with the analytic solution (3.7) that is obtained in the continuum limit. While both the discrete and the continuum versions of the CDPR lead to the same qualitative conclusions, the precise form of the stationary density is influenced by the fact that the simulations were carried out on a discrete chain. How to intuitively account for the surprisingly complex behaviour of the stationary density remains an open question. This might be related to the non-equilibrium nature of the stationary state (dynamics without detailed balance, even for $r = 0$).

3.2. Dynamics

Now, we complete this study by deriving the analytical solution for dynamical part $b(t, x)$ which satisfies

$$\partial_\tau b(\tau, x) = \partial_x^2 b(\tau, x) - \beta^3 x b(\tau, x) - \alpha^2 b(\tau, x) \tag{3.10}$$

[6] In the simple coagulation–diffusion process starting from an initially fully occupied lattice, this is the exact shape of the empty-interval probability $E_n(t)$ with a known time-dependent density $c = c(t)$ [14].

where time was rescaled according to $\tau := 2Dt$ and one also has the boundary and initial conditions

$$b(\tau, 0) = b(\tau, \infty) = 0, \quad b(0, x) = b_0(x) = 2E_0(x) - f(x). \tag{3.11}$$

In principle, one may solve this by using a Laplace transform $\bar{b}(s, x) = \int_0^\infty e^{-s\tau} b(\tau, x)$. This gives the equation

$$\partial_x^2 \bar{b}(s, x) - \beta^3 x \bar{b}(s, x) - (s + \alpha^2)\bar{b}(s, x) + b_0(x) = 0 \tag{3.12}$$

along with the boundary conditions $\bar{b}(s, 0) = \bar{b}(s, \infty) = 0$. This is the same type as equation (3.4). An analogous straightforward, but just a little tedious, calculation leads to

$$\begin{aligned}\bar{b}(s, x) = \frac{\pi}{\beta}\Bigg\{ &-\int_0^\infty \mathrm{d}Y\, b_0(Y)\mathrm{Ai}(\beta(Y+\mu) + s/\beta^2)\mathrm{Ai}(\beta(x+\mu) + s/\beta^2)\frac{\mathrm{Bi}(\beta\mu + s/\beta^2)}{\mathrm{Ai}(\beta\mu + s/\beta^2)} \\ &+ \int_x^\infty \mathrm{d}Y\, \mathrm{Ai}(\beta(Y+\mu) + s/\beta^2)\mathrm{Bi}(\beta(x+\mu) + s/\beta^2) \\ &+ \int_0^x \mathrm{d}Y\, \mathrm{Bi}(\beta(Y+\mu) + s/\beta^2)\mathrm{Ai}(\beta(x+\mu) + s/\beta^2)\Bigg\}. \end{aligned} \tag{3.13}$$

Generalizing [34], the inverse Laplace transform is now formally found from the poles of $\bar{b}(s, x)$, which arise via the zeroes of the Airy function, in the first term. The result is

$$b(t, x) = -\pi\beta \sum_{n=1}^\infty \int_0^\infty \mathrm{d}x'\, b(0, x')\mathrm{Ai}(\beta x' + a_n)\mathrm{Ai}(\beta x + a_n)\frac{\mathrm{Bi}(a_n)}{\mathrm{Ai}'(a_n)}\exp(-t(r + |a_n|\beta^2) \tag{3.14}$$

where a_n is the nth zero of the Airy function [1]. From this, one can read off the leading relaxation time $\tau_{\mathrm{rel}} = 1/(|a_1|\beta^2 + r)$, which is finite.

4. Inter-particle distribution function

In order to better appreciate the physical nature of the stationary state, we now study the properties of the *inter-particle distribution function* (IPDF), denoted here as $\mathcal{D}(x)$. On a discrete lattice $\mathcal{D}_n$ would be the probability that the nearest neighbour of a particle would be at a distance of n sites. In the continuum limit, this becomes $\mathcal{D}(x)$. The relation to the stationary empty-interval probability $E_{\mathrm{stat}}(x) = \frac{1}{2}f(x)$ is well-known [6]

$$\mathcal{D}(x) = \frac{1}{2\rho_{\mathrm{stat}}}\frac{\partial^2 f(x)}{\partial x^2} \tag{4.1}$$

with the stationary density ρ_{stat} found above. For the following discussion, we shall require the well-known expressions for $\mathcal{D}(x)$, as listed in table 2, for three paradigmatic systems; see, for example, [6] and the references therein for the computational details.

Clearly, the case (a) of uncorrelated particles will be an important test case for the study of the effects of a reset. Recall that this distribution is also obtained for a *reversible* coagulation–diffusion process with the extra reaction $A \to A + A$ [6], such that the stationary state is an equilibrium state. The second case (b) describes the correlations spontaneously generated during a coagulation–diffusion process $A + A \to A$. In these two cases, the parameter c denotes the average particle-density. It is known that, for an arbitrary initial condition in pure coagulation–diffusion, the system converges towards this distribution, with an explicitly known time-dependent concentration $c = c(t)$ [6, 14, 37]. Finally, case (c) gives the stationary distribution with an additional input of particles. For later comparisons, recall the asymptotic form $\mathcal{D}(x) \overset{x\to\infty}{\sim} x^{3/4}\exp\left(-\frac{2}{3}(\beta x)^{3/2}\right)$.

Table 2. The stationary empty-interval probability $E_{\rm stat}(x)$ and the corresponding IPDF $\mathcal{D}(x)$ for three types of systems: (a) uncorrelated particles, (b) coagulation–diffusion, and (c) with additional particle input. The distributions are characterized by the model parameters c and β.

	Type	$E_{\rm stat}(x)$	$\mathcal{D}(x)$
(a)	Uncorrelated	$\exp(-cx)$	$c\exp(-cx)$
(b)	Coagulation–diffusion	$\mathrm{erfc}(\frac{1}{2}\sqrt{\pi}\,cx)$	$\frac{1}{2}\pi c^2\, x\exp\left(-\frac{\pi}{4}c^2x^2\right)$
(c)	With particle-input	$\mathrm{Ai}(\beta x)/\mathrm{Ai}(0)$	$\beta^2 x\mathrm{Ai}(\beta x)/\lvert\mathrm{Ai}'(0)\rvert$

It is clear from table 2 that the three systems are clearly distinguished via their IPDFs for large interval sizes $x \to \infty$. This observation will become the central tool in our analysis of the IPDF with a reset.

4.1. IPDF without input

Using the previous expression (2.12) of the function $f(x)$, and the definition (4.1), the IPDF can be cast into a scaling form

$$\mathcal{D}(x) = \alpha D(\xi, v), \quad \xi := cx, \quad v := \frac{\alpha}{c} \tag{4.2}$$

with the explicit scaling function

$$D(\xi, v) = \frac{\alpha}{\rho_{\rm stat}}\left\{ e^{-v\xi} - F\left(\frac{\xi}{c}\right) + \frac{v}{2}\left[\int_0^\xi dY\, F\left(\frac{Y}{c}\right) e^{v(Y-\xi)} + \int_\xi^\infty dY\, F\left(\frac{Y}{c}\right) e^{v(\xi-Y)} \right.\right.$$
$$\left.\left. - \int_0^\infty dY\, F\left(\frac{Y}{c}\right) e^{-v(Y+\xi)}\right]\right\}. \tag{4.3}$$

Next, we shall discuss two specific examples. First, for a reset to uncorrelated particles with mean density c, one has $F(x) = e^{-cx} = e^{-\xi}$ and

$$D_{(a)}(\xi, v) = \frac{\exp(-v\xi) - \exp(-\xi)}{1 - v}. \tag{4.4}$$

Second, for a reset to a coagulation–diffusion configuration with density c, one has $F(x) = \mathrm{erfc}(\frac{1}{2}\sqrt{\pi}\,\xi)$, hence

$$D_{(b)}(\xi, v) = \frac{1}{2\,\mathrm{erfc}(v/\sqrt{\pi})}\left[e^{-\xi v}\left(1 + \mathrm{erf}\left(\xi\frac{\sqrt{\pi}}{2} - \frac{v}{\sqrt{\pi}}\right)\right) - e^{\xi v}\,\mathrm{erfc}\left(\xi\frac{\sqrt{\pi}}{2} + \frac{v}{\sqrt{\pi}}\right)\right].$$

These functions are displayed in figure 5. First, in the left panel, the scaling function $D_{(a)}$ is shown. For the simple coagulation–diffusion process under study here, one would expect, consulting table 2(b), a Gaussian shape of the IPDF. Clearly, this is no longer the case in the presence of a reset. Rather, one sees that, although $D(\xi) \stackrel{\xi\to 0}{\to} 0$ as it should be for a stationary IPDF [6], for larger intervals one has an exponential distribution, which is typical of a system of *uncorrelated* particles[7]. Furthermore, one observes that the effective density of particles in the large-ξ regime (which can be read off from the slope of $\ln D(\xi)$) depends in a non-trivial way on the scaling parameter $v = \alpha/c$. Namely, if $v > 1$, then the effective particle-density is simply unity, whereas if $v < 1$, then the effective particle-density is equal to v (it remains to be seen whether this kind of dynamical transition also occurs in different models). In conclusion,

[7] This is analogous to the finding of EM that the probability distribution of a random walk with reset is no longer Gaussian.

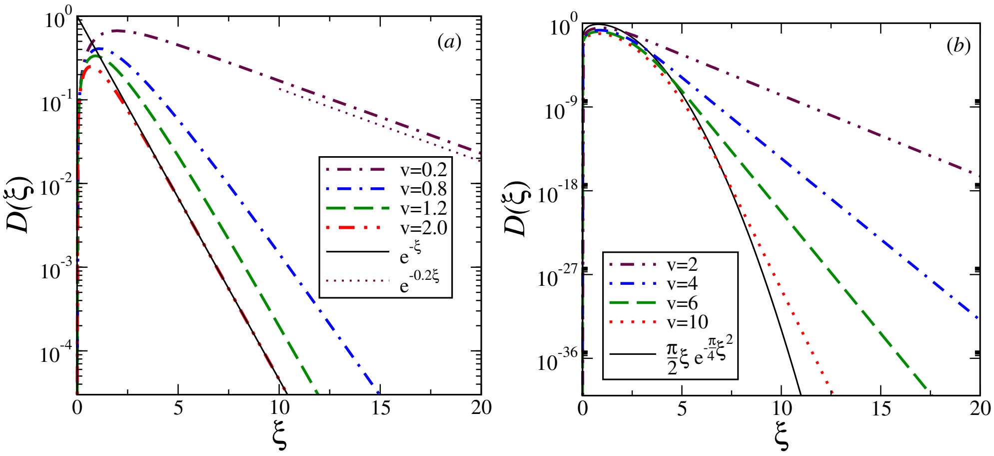

Figure 5. Stationary IPDF $D(\xi, v)$ of the coagulation–diffusion process, as a function of the scaled interval size ξ and several values of the scaling variable v. Left panel (a): reset distribution $F(x) = \exp(-cx)$; the IPDF equation (4.4) is shown for $v = [0.2, 0.8, 1.2, 2.0]$ from top to bottom. The distributions $e^{-\xi}$ and $e^{-0.2\xi}$ are also indicated. Right panel (b): reset distribution $F(x) = \text{erfc}(\frac{1}{2}\sqrt{\pi}\, cx)$; the IPDF equation (4.5) is shown with $v = [2, 4, 6, 10]$ from top to bottom. The distribution $\partial_\xi^2\, \text{erfc}(\frac{\sqrt{\pi}}{2}\xi) = \frac{\pi}{2}\xi \text{e}^{-\frac{\pi}{4}\xi^2}$ is also shown.

the behaviour of the IPDF at small scales is quite distinct from that seen at large scales. This is a consequence of the fact that the stationary state in the presence of a reset can no longer be described as an equilibrium state.

A further aspect of this become apparent if a different reset configuration is analysed, see the right panel of figure 5 with the scaling function $D_{(b)}$. Here, the reset is done to a configuration of particles as obtained from an usual coagulation–diffusion process, with the natural correlations corresponding to a given density c. At first sight, one might expect that the reset to such a correlated configuration should lead to these correlations being maintained for all interval sizes ξ. However, it can be seen from figure 5 that this is not so. Rather, in the stationary state, the correlated particle configurations only describe the actual *stationary* IPDF at small interval sizes ξ. At larger sizes, one again observes an effective distribution corresponding to uncorrelated particles.

Intuitively, the observation from figure 5(b) may be understood as follows: through the reset rate r, a further time scale $\tau_r \sim \alpha^{-2}$ is introduced, which in turn creates a new length scale $\xi_r \sim \alpha$. Between two reset events, the system has on average enough time to reconstitute its natural correlations up to scales $\xi \lesssim \xi_r$ but, since the resets are uncorrelated, beyond that scale its particles have become uncorrelated. The non-equilibrium nature of the stationary state manifests itself in strong short-distance correlations, as prescribed by the original dynamics, and in an uncorrelated behaviour at large distances.

4.2. IPDF with input

From the previous equation (3.6), the stationary (IPDF) is cast in the scaling form

$$\mathcal{D}(x) := \frac{1}{2\rho_{\text{stat}}}\frac{\partial^2 f(x)}{\partial x^2} = \frac{\beta^2}{\rho_{\text{stat}}} D(\beta x, c/\beta, \beta\mu) \tag{4.5}$$

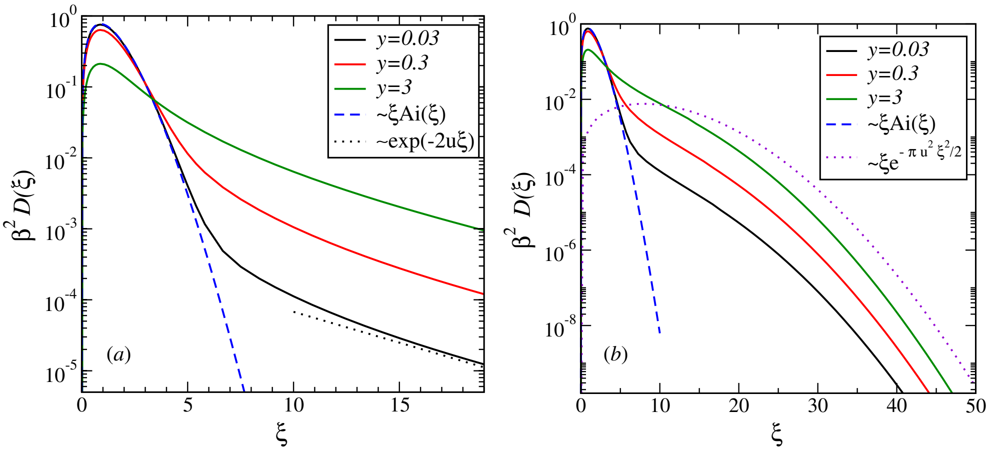

Figure 6. Stationary IPDF of the CDPR with input, for the case $u = 0.1$ and several values of y. Left panel (a): reset distribution $F(x) = \exp(-cx)$. The IPDFs for diffusion-coagulation with input, both for input without a reset and for independent particles, are also shown. Right panel (b): reset distribution $F(x) = \mathrm{erfc}(\frac{1}{2}\sqrt{\pi}\,cx)$. The IPDFs for diffusion-coagulation with input and for diffusion-coagulation are also shown.

with the three-variable scaling function

$$
\begin{aligned}
D(\xi, u, y) = \frac{(\xi+y)\mathrm{Ai}(\xi+y)}{\mathrm{Ai}(y)} &- \pi y(\xi+y)\mathrm{Ai}(\xi+y)\frac{\mathrm{Bi}(y)}{\mathrm{Ai}(y)}\int_0^\infty \mathrm{d}Y\, F(uY/c)\mathrm{Ai}(Y+y) \\
&+ \pi y(\xi+y)\mathrm{Bi}(\xi+y)\int_\xi^\infty \mathrm{d}Y\, F(uY/c)\mathrm{Ai}(Y+y) \\
&+ \pi y(\xi+y)\mathrm{Ai}(\xi+y)\int_0^\xi \mathrm{d}Y\, F(uY/c)\mathrm{Bi}(Y+y) \\
&+ \pi y F(u\xi/c)[\mathrm{Bi}(\xi+y)\mathrm{Ai}'(\xi+y) - \mathrm{Bi}'(\xi+y)\mathrm{Ai}(\xi+y)] \qquad (4.6)
\end{aligned}
$$

and the scaling variables $\xi := \beta x$, $u := c/\beta$ and $y := \beta\mu$.

The consequences of the reset are illustrated in figure 6. First, for a reset to uncorrelated particles, the behaviour seen in the left panel (figure 6(a)) is qualitatively the same as that seen above in the case without input. At short interval sizes, the system has enough time between two resets to build up its natural correlations, so that the shape of the IPDF is essentially given by the Airy function (see table 2) and its stretched-exponential form. For larger sizes, the particles become uncorrelated and the IPDF goes over to a simple exponential.

A similar pattern is seen when resetting to configurations of correlated particles. In the right panel (figure 6(b)), the IPDF for a reset to a configuration of simple diffusion-coagulation with mean density c is shown. With respect to the previous situation, the IPDF of the reset distribution $F(\xi)$ falls off more rapidly for $\xi \gg 1$ than the 'natural' one of the underlying process. Yet, we see that the reset rate r once again sets a time scale such that, for sufficiently small interval sizes, the distribution of the empty intervals is the natural one of diffusion-coagulation with input and goes over to the one put in by the reset for larger intervals[8].

In any case, these examples illustrate the subtle nature of the stationary states in simple particle-reaction models with a stochastic reset. The main new feature is a new scale set by the

[8] We did not detect any evidence that, for extremely large values of ξ, the IPDF would cross over to a simple exponential form.

reset rate $r > 0$, such that, at sufficiently small length scales, the 'natural' correlations of the dynamics dominate whereas, at larger length scales, those of the reset configurations become dominant.

5. Conclusions

Our analysis of the effects of a stochastic reset provides an alternative route to better appreciate the consequences of the breaking of detailed balance. This breaking is required to obtain non-equilibrium stationary states. In the present work, we have studied how the properties of a simple reaction-diffusion model are modified through the introduction of a stochastic reset. This was achieved by identifying a new member in the class of models, which may be solved exactly through the 'empty-interval method'.

A particular bonus of this technique is that it provides a very direct access to the distribution of the distances between particles. In this way, we have seen that: (i) the model's behaviour is not much affected by the reset at short length scales, but that (ii) it is profoundly altered at larger scales. The coexistence of at least two kinds of correlations at different length scales should be identified as the main mechanism that drives the system to a new non-equilibrium stationary state.

Comparison of our analytical results with the Monte Carlo simulations permit us to identify how to set up analogous studies in different many-body problems and/or networks, where exact analytical results may not be so readily available.

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Appendix. Remarks on the Monte Carlo simulations

In order to further illustrate the proper choice of a microscopic model of the CDPR, which for continuous time would be described by (1.2), we compare two different choices for the transition rates in a Monte Carlo simulation:

- *Method 1 :* as already defined in section 1 in the main text. We insist that the probability $\mathcal{P}_r$ chosen for the reset guarantees a reset with probability r per sweep.
- *Method 2 :* for each sweep of $\mathcal{N}$ individual Monte Carlo steps, the move to be carried out it is globally chosen for all particles: either $\mathcal{N}$ usual coagulation–diffusion steps are selected with probability $2D/(2D + r)$, or else a global reset is chosen, with probability $r/(2D + r)$.

In figure A1, the results for the time-dependent density $\rho(t)$ of these choices of the dynamics are shown, for a periodic chain with $\mathcal{N} = 512$ sites, $D = \frac{1}{2}$, a reset to uncorrelated particles with mean density $p = 0.5$ and several values of r. Comparison with the exact result, derived in section 2, shows that while the data obtained from method 1 perfectly agree, there is no agreement with those from method 2.

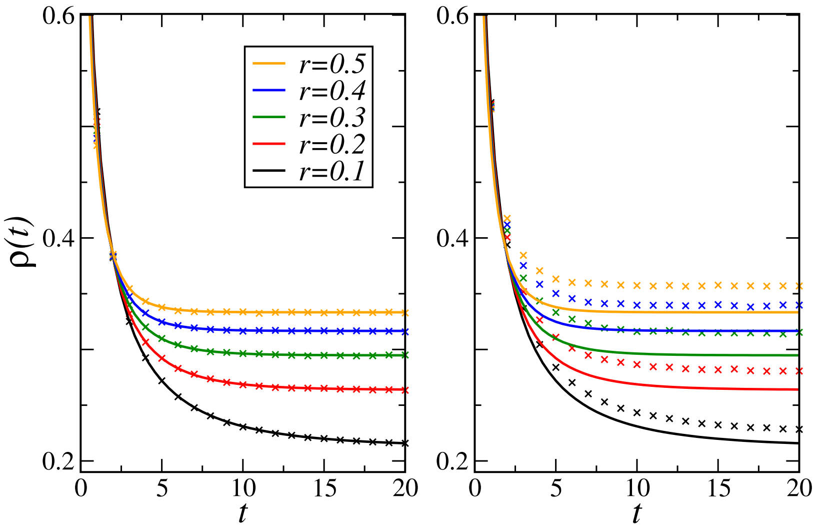

Figure A1. Time-dependent particle-density $\rho(t)$ in two distinct Monte Carlo simulations on a periodic chain with $\mathcal{N} = 512$ sites. Left panel: result of method 1, right panel: results of method 2. The parameters used are $D = 1/2$, $p = 0.5$ and $r = [0.1, 0.2, 0.3, 0.4, 0.5]$ from bottom to top. Initially the system was entirely filled. The full lines give the exact analytical result, see section 2.

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