Universality Classes of Critical Phenomena

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Abstract: *A brief review is given on universality classes of critical phenomena. Similarities and differences between equilibrium and nonequilibrium critical phenomena are discussed.*

1. Introduction

During the past decades, the area of equilibrium critical phenomena in phase transitions has been one of the most active research areas in the condensed matter physics. Especially the universality of critical phenomena has drawn great interests from many physicists. Physical systems usually involve a very large number of parameters and their interaction Hamiltonians are very complicated. However, there exist wide classes of physically different systems showing identical critical behavior[1]. This indicates that the critical behavior does not depend on microscopic details of interaction Hamiltonians, but only on some general features of the degrees of freedom. Critical behaviors of physical systems can be classified into universality classes characterized by various fractal dimensions associated with geometrical patterns at criticality. The values of these fractal dimensions depend on the spatial dimensionality and the symmetry properties between ground states of a given system.

In late sixties, the universality of critical phenomena had been confirmed by many experimental results[2]. The concept of the universality led to Widom's scaling postulate[3], Kadanoff's block spin formalism[4], and finally materialized in the renormalization group theory[5]. Recently the conformal field theory where one assumes the invariance of critical systems under the conformal transformations explored enormously many integrable systems of different universality classes in two dimensional critical phenomena in equilibrium[6].

Critical phenomena in equilibrium systems have been studied extensively and most of their critical properties have been uncovered. However, only a few universality classes are known in nonequilibrium systems so far. In this article, I will briefly review the universality classes in equilibrium and nonequilibrium systems and discuss similarities and differences between equilibrium and nonequilibrium critical phenomena by comparing q-state equilibrium Potts phase transitions[7] and q-state nonequilibrium absorbing phase transitions[8].

2. Equilibrium critical phenomena

Equilibrium models are defined by Hamiltonians and their critical properties are determined by the symmetry properties of their Hamiltonians. The Ising model is one of the mostly investigated equilibrium models and its Hamiltonian is

$$H = -J \sum_{<i,j>} S_i S_j - h \sum_i S_i \qquad (1)$$

where J is the coupling constant, h is the magnetic field, and S_i is a spin variable at site i. The first summation is over nearest-neighboring sites, i and j, and the spin variable S can take either $+\frac{1}{2}$ or $-\frac{1}{2}$.

At $h = 0$, this model exhibits a continuous transition from the disordered phase into the ferromagnetically ordered phase as the temperature T is lowered. At the transition ($T = T_c$), the system becomes critical, i.e. the correlation length ξ diverges as $\xi \sim t^{-\nu}$ where t is the reduced temperature: $t = |T - T_c|/T_c$. The specific heat C and the magnetic susceptibility χ also diverge as

$C \sim t^{-\alpha}$ and $\chi \sim t^{-\gamma}$. The order parameter m (magnetization) vanishes at criticality with $m \sim t^{\beta}$. These critical exponents, ν, α, β and γ, are simply related to the fractal dimensions of geometrical patterns at criticality by the scaling postulate. The fractal dimensions, d_ℓ and d_a, associated with coast lines and areas of ferromagnetic domains are known exactly in two dimensions, $d_\ell = 1$ and $d_a = \frac{15}{8}$. When the magnetic field h is turned on, the system becomes noncritical throughout all temperature range.

The Ising system is invariant under $S \to -S$ transformation when $h = 0$. So the ground states have two-fold degeneracy. Generalizations of the Ising model to models with higher symmetry are diverse; the n-vector model with rotational symmetry[9], the q-state clock model with cyclic symmetry[10], and the q-state Potts model with permutation symmetry[7]. In this article, I will focus on the Potts model.

The Potts Hamiltonian is given by

$$H = -J \sum_{<i,j>} \delta_{\sigma_i,\sigma_j} \tag{2}$$

where σ_i is a spin variable at site i which can assume q discrete values ($\sigma_i = 0, 1, \cdots, q-1$). The Hamiltonian is invariant under the permutations of q spin states and the ferromagnetic ground states have q-fold degeneracy. When $q = 2$, this model becomes equivalent to the Ising model.

The order-disorder transition of the q-state Potts model is continuous for $q \leq 4$ in two dimensions. This model can be mapped into the Gaussian model through which the fractal dimensions can be obtained exactly for general q in two dimensions[11]. These fractal dimensions vary with the value of q, so the Potts model includes three different universality classes, i.e. Ising ($q = 2$), 3-state Potts ($q = 3$), and Ashkin-Teller ($q = 4$) universality class[12].

Reformulation of the Potts model as the random cluster model by Fortuin and Kasteleyn[13] brings in the physical meaning of the $q \to 1$ limit, which turns out to be equivalent to the percolation problem. Therefore the percolation problem can be viewed as a model with a non-degenerated ground state. The percolation problem has not been solved exactly by itself but the relation with the q-state Potts model allows one to calculate its critical exponents analytically by taking the $q \to 1$ limit to the q-dependent fractal dimensions of the Potts model.

3. Nonequilibrium critical phenomena

Many nonequilibrium models show continuous phase transitions similar to equilibrium models. In fact, nonequilibrium models supply much richer critical behavior because the evolving dynamics do not require the detailed balance. Therefore the universality classes of nonequilibrium critical phenomena would be much more diverse and would be governed by various symmetric properties of the evolution dynamics.

Phase transitions in nonequilibrium systems occur between distinct nonequilibrium steady states. For example, consider the Ising model in contact with two thermal reservoirs at different temperatures[14, 15]. The competition of two dynamics at different temperatures drives the system in nonequilibrium steady states. There is a continuous phase transition between the ordered and disordered steady states, but the universality class of this phase transition depends on the type of the dynamics. When the spin-flip Glauber dynamics[16] are employed, its critical property does not differ from the equilibrium case[14]. In contrast, when the order-parameter conserving Kawasaki dynamics[17] are employed, a new universality class emerges[15]. It is generally known that the conservation laws in dynamics play an important role in determining the universality class in nonequilibrium.

Another example of nonequilibrium phase transitions is the absorbing phase transition. In this case, there exist some absorbing states in the configurational phase space. If the system gets into one of the absorbing states by the evolution dynamics, then the system is trapped inside of the absorbing states and no further dynamics proceed. By controlling an external parameter, one can observe a continuous phase transition from a steady-state active phase into an inactive absorbing phase.

The simplest model exhibiting an absorbing phase transition is the contact process[18] which describes an epidemic process without immunization. Start with randomly distributed particles on a lattice. A particle can create a new particle with probability p in one of its neighboring sites when it is empty or desorb from the lattice with probability $1 - p$. The former process can be interpreted as the infection of diseases in neighborhood and the latter as the spontaneous healing. In this model, the vacuum state is the only absorbing state and the external parameter is p. As the value of p is lowered, a continuous transition occurs at finite probability p_c from the active phase into the inactive absorbing phase. The order parameter of this model is the density of particles which vanishes at the transition with $\rho \sim (p - p_c)^{\beta}$. The value of the order-parameter exponent β can be obtained by numerical simulations and series expansion techniques, which turns out to be equivalent to that of the directed percolation (DP) problem[19].

Many other models such as the monomer dimer model[20], the interacting monomer monomer model[21], the branching annihilating random walks with an odd number of offsprings[22] are also shown to belong to the DP universality class. A common feature of these models is that they exhibit a phase transition from an active phase into an inactive phase of a *single* absorbing state.

This leads to the so-called "DP conjecture"[23] which depicts that models exhibiting a continuous transition to a single absorbing state generally belong to the DP universality class.

The universality class for models with a single absorbing state is well established. But few studies have been made for models with more than one absorbing states. Recently Jensen and Dickman[24] have studied some nonequilibrium models with infinitely many absorbing states, the pair contact process and the reaction dimer model. Both models are shown, rather surprisingly, again in the DP universality class. This result might mislead the reader that the number of absorbing states is not relevant to the universality class of the absorbing phase transitions. This shows a sharp contrast to the case of equilibrium critical phenomena where the number of ground states plays a crucial role. The symmetry between ground states governs the critical behavior rather than the number of ground states. In the models studied by Jensen and Dickman, the infinitely many absorbing states are not equivalent probabilistically. So it is important to study a model with multiple equivalent absorbing states.

A few models have appeared in the literature which have two equivalent absorbing states. Those are the models of probabilistic cellular automata introduced by Grassberger et al.[25], nonequilibrium kinetic Ising models with two different dynamics[26], and the interacting monomer-dimer model[27]. Critical behaviors of these models are different from DP but form a new non-DP universality class. This result confirms our assertion that the symmetry between absorbing states is relevant to the universality class. In the kink (or domain-wall) representation, these models have a common feature that the evolving dynamics conserve the number of kinks modulo 2. Like the other nonequilibrium critical phenomena, one may conclude that the conservation law also plays an important role in the absorbing phase transitions.

Recently, I studied various models with two *inequivalent* absorbing states but still with the evolving dynamics conserving the number of kinks modulo 2[28, 8]. This can be done by introducing a symmetry breaking field in the above three models. Numerical simulations show that there still exists a continuous absorbing transition and its critical behavior goes back to the DP universality class. This result clearly shows that the symmetry between absorbing states is much more important than the conservation laws in determining the universality class of the absorbing transitions. Very recently, Hinrichsen[29] introduced models with two equivalent absorbing states without conservation laws in its dynamics and showed that these models belong to the same non-DP universality class as in the models without a symmetry breaking field. One can conclude that the symmetry between absorbing states is indeed the only essential property in the absorbing transitions.

4. Similarities and differences

Nonequilibrium and equilibrium critical phenomena have some common aspects and also differences. In the absorbing phase transitions, they are quite similar. The symmetry between ground (or absorbing) states governs critical behavior. However, general nonequilibrium critical phenomena without absorbing states depend on the conservation laws. The absorbing phase transition is a special limit of general nonequilibrium phase transitions between distinct steady states. Therefore there may be more general statements about the universality classes of general nonequilibrium phase transitions. It may be also worthy to generalize the above models to models with higher symmetry.

It is well known that two dimensional equilibrium models are related to one dimensional kinetic models via transfer matrix formalism[30]. An extra space dimension is interpreted as time dimension in one dimensional kinetic model. One can write down the evolution operator of the kinetic model corresponding to a given two dimensional equilibrium model. This evolution operator is Hermitian for the equilibrium model. Nonequilibrium models in one dimension can be obtained by modifying the above evolution operator in a non-Hermitian form, i.e. breaking the detailed balance. Then equilibrium models in d dimensions and nonequilibrium models in $d-1$ dimensions can be directly compared.

It is interesting to compare the equilibrium Ising model ($q = 2$ Potts model) with nonequilibrium models with two equivalent absorbing states. Actually the nonequilibrium kinetic Ising model with two different dynamics[26] mentioned in the previous section is a special case of general nonequilibrium Ising model. In this model, time invariance is broken completely and its dynamics favor one time direction over the other. By adding a directional sense in time direction to the equilibrium Ising model, one can see the crossover from the equilibrium Ising universality class to the nonequilibrium non-DP universality class.

Similar things happen for the models with a single absorbing state which belong to the DP universality class. The percolation problem is equivalent to the $q \to 1$ limit of the q-state Potts model. The DP problem is defined by adding a directional sense to the percolation problem. Similarly, one can define nonequilibrium models with q absorbing states by adding a directional sense to the q-sate Potts model. Both models have permutation symmetry between q ground (or absorbing) states, so I will call these nonequilibrium models as the q-state directional Potts model. It will be interesting to set up and investigate these new models. Of course, other types of generalized models with rotation or cyclic symmetry are also interesting.

However, there is a difference in connecting models with different values of q. Crossover from the non-DP to the DP universality class is obtained by introducing a symmetry breaking field. But the Ising phase transition disappears when the magnetic field is applied to the equilibrium model. The percolation universality class appears only through the random cluster formulation of the Potts model.

Therefore the analogy between the q-state Potts model and the q-state directional Potts model is not complete. These similarities and differences should be further investigated in future.

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