Unconventional Entropy Production in the Presence of Momentum-dependent Forces

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We investigate the unconventional nature of entropy production (EP) in nonequilibrium systems with odd-parity variables that change signs under time reversal. We consider the Brownian motion of a particle in contact with a heat reservoir, where the particle's momentum is an odd-parity variable. In the presence of an *external* momentum-dependent force, the EP transferred to the environment is found to be *not* equivalent to the usual reservoir entropy change due to heat transfer. An additional unconventional contribution to the EP, which is crucial for maintaining the non-negativity of the (average) total EP enforced by the second law of thermodynamics, appears. A few examples are considered to elucidate the novel nature of the EP. We also discuss detailed balance conditions with a momentum-dependent force.

I. INTRODUCTION

Recent studies on nonequilibrium (NEQ) fluctuations were motivated by the discovery of the fluctuation theorems (FT's) [1–5]. The FT's were at first regarded as new identities or relations governing thermally-fluctuating quantities in various NEQ dynamics, deterministic or stochastic. More recent studies, however, have revealed many interesting properties beyond simple relations, which include excess and housekeeping (or nonadiabatic and adiabatic) contributions to the entropy production (EP) [6–11], modifications of FT's [12–15], the information entropy [16,17], multiple dynamical transitions [18], and the persistent initial-memory effect [19,20].

Most of the studies so far have dealt with stochastic systems where state variables have even parities (do not change signs) under time reversal. This is mainly due to their technical simplicity and might also be due to a naive presumption that the generalization to a case including odd-parity variables might be straightforward. However, systems with odd-parity variables have recently been recognized to exhibit a fundamental difference from those with even-parity variables only. For example, the housekeeping EP no longer satisfies the FT and can be divided into two parts characterized by the breakage of detailed balance (DB) and the parity asymmetry of the steady-state probability distribution function [10,11].

In this study, we report another fundamental difference, which is associated with the nature of the EP in systems with both even- and odd-parity variables, from those without odd-parity variables. To be specific, we consider an external momentum-dependent driving force in underdamped Brownian motion. Typical examples are velocity-dependent forces in active matter systems [21] such as granular particles under vibration [22], interacting molecular motors [23], and active Brownian particles [24, 25]. Simpler examples are the Lorentz force on a charged particle in a uniform magnetic field and an additional dissipative force through a feedback process in a molecular refrigerator (cold damping) [17,23,26].

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For a system in contact with a single thermal reservoir at temperature T, the conventional EP in the NEQ steady state can be characterized by an incessant environmental EP mediated solely by the heat dissipation Q into the reservoir such that $\Delta S_{env} = \Delta S_{res} = Q/T$. However, we find that this relation should be modified in the presence of external momentum-dependent forces and an extra environmental EP, ΔS_{uc} (unconventional EP), yielding $\Delta S_{env} = \Delta S_{res} + \Delta S_{uc}$, appears. In some cases, we can explicitly show that ΔS_{res} is negative by itself and that the positivity of ΔS_{env} is restored by adding ΔS_{uc}. Therefore, the existence of ΔS_{uc} is crucial for the validity of the second law of thermodynamics ($\Delta S_{env} \geq 0$ in the steady state) [27]. A similar extra contribution to the environmental EP has been considered in Ref. 24 for a one-dimensional system with a simple form of a momentum-dependent force.

II. MODEL

Consider the driven Brownian motion of a particle of mass m in state space $\mathbf{q} = (\mathbf{x}, \mathbf{p})$, where $\mathbf{x}$ and $\mathbf{p}$ are the position and the momentum vectors in d space dimensions, respectively. The stochastic differential equation reads

$$\frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{m} , \quad \frac{d\mathbf{p}}{dt} = \mathbf{f}(\mathbf{q}; \lambda(t)) - \mathsf{G} \cdot \frac{\mathbf{p}}{m} + \boldsymbol{\xi}(t) , \quad (1)$$

where G is a dissipation matrix and a dot ($\cdot$) between matrices (or vectors) denotes a contraction (inner product). The Gaussian noise vector $\boldsymbol{\xi}(t)$ satisfies $\langle \boldsymbol{\xi}(t)\boldsymbol{\xi}^{t}(t')\rangle = 2\mathsf{D}\delta(t - t')$, with the superscript 't' denoting 'transpose'. The diffusion matrix D is positive definite and symmetric. The last two terms of the above equation describe *thermal forces* exerted by heat reservoirs. With a single heat reservoir at temperature T, the Einstein relation holds as $\mathsf{D} = T\mathsf{G}$. The most general form of the force $\mathbf{f}$ is considered, which may depend on both the position and the momentum, as well as on a time-dependent protocol $\lambda(t)$. We note that previous studies [24] have only focused on the case where $\mathbf{f}$ is given by the sum of two terms, one of which depends only on position and the other on momentum.

The corresponding Kramers equation for the probability distribution function (PDF), $\rho(\mathbf{q}, t)$, reads

$$\partial_t \rho = -\mathsf{L}\rho , \quad (2)$$

with the evolution operator L being given by

$$\mathsf{L}(\mathbf{q}) = \partial_{\mathbf{x}} \cdot \frac{\mathbf{p}}{m} + \partial_{\mathbf{p}} \cdot \left(\mathbf{f} - \mathsf{G} \cdot \frac{\mathbf{p}}{m} - \mathsf{D} \cdot \partial_{\mathbf{p}} \right) . \quad (3)$$

The formal solution can be written in terms of a path integral for $0 \leq t \leq \tau$ as

$$\rho(\mathbf{q}, \tau) = \int d\mathbf{q}_0 \int D[\mathbf{q}(t)] \, \Pi[\mathbf{q}(t); \lambda(t)] \, \rho(\mathbf{q}_0) , \quad (4)$$

with the initial PDF $\rho(\mathbf{q}_0)$. The kernel $\Pi[\mathbf{q}(t); \lambda(t)]$ is the conditional path probability density of the system evolving along a given path $\{\mathbf{q}(t)\}$ for $0 \leq t \leq \tau$, starting at $\mathbf{q}(0) = \mathbf{q}_0$. The integration $\int D[\mathbf{q}(t)]$ is over all paths for given $\mathbf{q}_0$ and $\mathbf{q}(\tau)$. The probability density to find a path $\{\mathbf{q}(t)\}$ is given by $P[\mathbf{q}(t); \lambda(t)] = \Pi[\mathbf{q}(t); \lambda(t)] \, \rho(\mathbf{q}_0)$.

III. IRREVERSIBILITY

Entropy production measures the irreversibility of a process. To define irreversibility, we first consider the time-reverse process which should be governed by the equation of motion equivalent to that of the original (time-forward) process in Eq. (1):

$$\frac{d\bar{\mathbf{x}}}{d\bar{t}} = \frac{\bar{\mathbf{p}}}{m} , \quad \frac{d\bar{\mathbf{p}}}{d\bar{t}} = \mathbf{f}(\bar{\mathbf{q}}; \bar{\lambda}(\bar{t})) - \mathsf{G} \cdot \frac{\bar{\mathbf{p}}}{m} + \boldsymbol{\xi}(\bar{t}) , \quad (5)$$

where all variables in the time-reverse process are denoted by overbars. The time-dependent protocol function $\bar{\lambda}$ is set by requiring $\bar{\lambda}(\bar{t}) = \lambda(t)$.

For each (time-forward) path $\{\mathbf{q}(t)\}$ ($0 \leq t \leq \tau$), one can consider the corresponding time-reverse path $\{\bar{\mathbf{q}}(\bar{t})\}$ for $0 \leq \bar{t} \leq \tau$ with $\bar{t} = \tau - t$ and $\bar{\mathbf{q}}(\bar{t}) = \varepsilon\mathbf{q}(t)$, with parity $\varepsilon = 1$ for position $\mathbf{x}$ (even parity) and $\varepsilon = -1$ for momentum $\mathbf{p}$ (odd parity). Note that the odd parity for momentum is automatically enforced by identifying $\bar{\mathbf{x}}(\bar{t}) = \mathbf{x}(t)$ with $\bar{t} = \tau - t$ in Eq. (5). Irreversibility (or the total EP, ΔS_{tot}) for a given path $\{\mathbf{q}(t)\}$ is defined by

$$\Delta S_{tot}[\mathbf{q}(t)] = \ln \frac{\Pi[\mathbf{q}(t); \lambda(t)]\rho(\mathbf{q}_0)}{\bar{\Pi}[\bar{\mathbf{q}}(\bar{t}); \bar{\lambda}(\bar{t})]\bar{\rho}(\bar{\mathbf{q}}_0)} = \ln \frac{P[\mathbf{q}(t); \lambda(t)]}{\bar{P}[\bar{\mathbf{q}}(\bar{t}); \bar{\lambda}(\bar{t})]},$$
$$(6)$$

where $\bar{\Pi}[\bar{\mathbf{q}}(\bar{t}); \bar{\lambda}(\bar{t})]$ is the conditional path probability density for the time-reverse path $\{\bar{\mathbf{q}}(\bar{t})\}$ and its initial PDF is given as $\bar{\rho}(\bar{\mathbf{q}}_0) = \rho(\mathbf{q}(\tau))$.

For a *reversible* path, the path probabilities P and $\bar{P}$ are the same by definition, and no EP is expected. Deviation from this accounts for the irreversibility of a path. The usual (average) environmental EP, $\langle \Delta S_{env} \rangle$, is obtained by averaging over all possible paths and the initial PDF. The usual (average) total EP, $\langle \Delta S_{tot} \rangle$, is obtained by averaging over all possible paths. A simple probability normalization condition results in the integral FT expressed as $\langle e^{-\Delta S_{tot}} \rangle = 1$ [8,9]. With the Jensen's inequality, it leads to $\langle \Delta S_{tot} \rangle \geq 0$, which proves the second law of thermodynamics.

The total EP can be divided into the system entropy change and the environmental EP as $\Delta S_{tot} = \Delta S_{sys} + \Delta S_{env}$, where $\Delta S_{sys} = -\ln \rho(\mathbf{q}(\tau)) + \ln \rho(\mathbf{q}_0)$ is given by the Shannon entropy difference. Thus, from Eq. (6), the environmental EP $\Delta S_{env}[\mathbf{q}(t)]$ is simply the logarithm of the ratio of the two conditional path-probability densities.

Discretizing time such that $t_i = i\Delta t$ for $i = 0,\ldots,N$ with the interval $\Delta t = \tau/N$ for large N is useful. A path is now represented by a discrete sequence $\{\mathbf{q}(t)\} = \{\mathbf{q}_0,\ldots,\mathbf{q}_i,\ldots,\mathbf{q}_N\}$, where $\mathbf{q}_i = \mathbf{q}(t_i)$. The Markovian property of dynamics allows us to write

$$\Pi[\mathbf{q}(t); \lambda(t)] = \prod_{i=1}^{N} \Gamma(\mathbf{q}_i|\mathbf{q}_{i-1}; \lambda_{i-1}), \qquad (7)$$

where $\Gamma(\mathbf{q}_i|\mathbf{q}_{i-1}; \lambda_{i-1}) = \langle \mathbf{q}_i|e^{-\Delta t \mathsf{L}}|\mathbf{q}_{i-1}\rangle$ is the conditional probability for the system at $(\mathbf{q}_i, t_i)$, starting from $(\mathbf{q}_{i-1}, t_{i-1})$ with $\lambda_{i-1} = \lambda(t_{i-1})$. Using a quantum-mechanical formalism for the non-Hermitian evolution operator L, we find

$$\Gamma(\mathbf{q}_i|\mathbf{q}_{i-1}; \lambda_{i-1}) = \frac{\delta(\mathbf{x}_i - \mathbf{x}_{i-1} - \Delta t\, \mathbf{p}_i^{(\alpha)}/m)}{|\det(4\pi\Delta t\mathsf{D})|^{d/2}} \times \exp\left\{-\frac{\Delta t}{4}\mathbf{h}_i^{(\alpha)} \cdot \mathsf{D}^{-1} \cdot \mathbf{h}_i^{(\alpha)} + \alpha\Delta t\left(\frac{\mathrm{Tr}\mathsf{G}}{m} - \partial_{\mathbf{p}^{(\alpha)}} \cdot \mathbf{f}_i^{(\alpha)}\right)\right\}, \qquad (8)$$

where

$$\mathbf{h}_i^{(\alpha)} = \frac{\mathbf{p}_i - \mathbf{p}_{i-1}}{\Delta t} + \mathsf{G} \cdot \frac{\mathbf{p}_i^{(\alpha)}}{m} - \mathbf{f}_i^{(\alpha)}, \qquad (9)$$

and $\mathbf{p}_i^{(\alpha)} = (1-\alpha)\mathbf{p}_{i-1} + \alpha\mathbf{p}_i$ for $0 \le \alpha \le 1$ represents an intermediate value of $\mathbf{p}$ during time interval $[t_{i-1}, t_i]$. In principle, α can be arbitrary, but should not affect physical observables like EPs for the large N (small Δt) limit. The force $\mathbf{f}_i^{(\alpha)}$ is defined in the same manner.

It is more convenient to consider the EP during an infinitesimal time interval $[t, t+dt]$ as

$$\mathrm{d}S_{env} = \ln\frac{\Gamma(\mathbf{q}', t+\mathrm{d}t|\mathbf{q}, t; \lambda(t))}{\Gamma(\varepsilon\mathbf{q}, t+\mathrm{d}t|\varepsilon\mathbf{q}', t; \lambda(t))}, \qquad (10)$$

where the conditional probability for the time-reverse path $(\bar{\mathbf{q}}' \to \bar{\mathbf{q}})$ is expressed in terms of that for the time-forward path $(\varepsilon\mathbf{q}' \to \varepsilon\mathbf{q})$ in the denominator. Then, a straightforward algebra yields

$$\frac{\mathrm{d}S_{env}}{\mathrm{d}t} = -\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \mathbf{f}^{\mathrm{rev}}\right) \cdot \mathsf{D}^{-1} \cdot \left(\mathsf{G} \cdot \frac{\mathbf{p}}{m} - \mathbf{f}^{\mathrm{ir}}\right) - \partial_{\mathbf{p}} \cdot \mathbf{f}^{\mathrm{rev}}, \qquad (11)$$

where the force is divided into the reversible and the irreversible parts, $\mathbf{f}(\mathbf{q}) = \mathbf{f}^{\mathrm{rev}}(\mathbf{q}) + \mathbf{f}^{\mathrm{ir}}(\mathbf{q})$, such that $\mathbf{f}^{\mathrm{rev}}(\mathbf{q}) = \mathbf{f}^{\mathrm{rev}}(\varepsilon\mathbf{q})$ and $\mathbf{f}^{\mathrm{ir}}(\mathbf{q}) = -\mathbf{f}^{\mathrm{ir}}(\varepsilon\mathbf{q})$ [28]. All quantities in the above equation should be evaluated at the midpoint ($\alpha = 1/2$; Stratonovich convention). We note that the resulting expression is unique with midpoint values independent of the choices of the discretization schemes (*i.e.*, any α for the time-forward path and any $\bar{\alpha}$ for the time-reverse path) [29].

IV. CURRENTS

It is illuminating to consider the average EP in terms of currents $\mathbf{j}_{\mathbf{q}}$ defined by the continuity equation as

$$\partial_t \rho = -\partial_{\mathbf{q}} \cdot \mathbf{j}_{\mathbf{q}} = -\mathsf{L}\rho, \qquad (12)$$

where $\mathbf{j}_{\mathbf{q}}$ can be obtained from the evolution operator L in Eq. (3). We also divide the currents into the reversible and the irreversible parts [28] as

$$\mathbf{j}_{\mathbf{p}}^{\mathrm{rev}} = \mathbf{f}^{\mathrm{rev}}(\mathbf{q})\rho(\mathbf{q}), \qquad (13)$$

$$\mathbf{j}_{\mathbf{p}}^{\mathrm{ir}} = \left(\mathbf{f}^{\mathrm{ir}}(\mathbf{q}) - \mathsf{G} \cdot \frac{\mathbf{p}}{m} - \mathsf{D} \cdot \partial_{\mathbf{p}}\right)\rho(\mathbf{q}), \qquad (14)$$

with

$$\mathbf{j}_{\mathbf{x}}^{\mathrm{rev}} = \frac{\mathbf{p}}{m}\rho(\mathbf{q}) \qquad \text{and} \qquad \mathbf{j}_{\mathbf{x}}^{\mathrm{ir}} = \mathbf{0}. \qquad (15)$$

Showing that

$$\left\langle\frac{\mathrm{d}S_{sys}}{\mathrm{d}t}\right\rangle = -\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\mathbf{q}\, \rho\ln\rho = -\int \mathrm{d}\mathbf{q}\, \frac{\mathbf{j}_{\mathbf{p}} \cdot \partial_{\mathbf{p}}\rho}{\rho} \quad (16)$$

is trivial. Averaging the environmental EP in Eq. (11), we need to evaluate the two-time correlators at time t and $t+\mathrm{d}t$, such as $\langle A(\mathbf{q}')B(\mathbf{q})\rangle = \int \mathrm{d}\mathbf{q}'\mathrm{d}\mathbf{q}\, A(\mathbf{q}')\Gamma(\mathbf{q}', t+\mathrm{d}t|\mathbf{q}, t)B(\mathbf{q})\rho(\mathbf{q})$. After some algebra, we find the average total EP rate as

$$\left\langle\frac{\mathrm{d}S_{tot}}{\mathrm{d}t}\right\rangle = \int \mathrm{d}\mathbf{q}\, \frac{\mathbf{j}_{\mathbf{p}}^{\mathrm{ir}} \cdot \mathsf{D}^{-1} \cdot \mathbf{j}_{\mathbf{p}}^{\mathrm{ir}}}{\rho} \ge 0, \qquad (17)$$

where the inequality (second law of thermodynamics) comes from the positive-definiteness of D [30]. Only the irreversible part $\mathbf{j}_{\mathbf{p}}^{\mathrm{ir}}$ of the current is a single measure for the irreversibility. In the overdamped limit, $\langle\mathbf{j}_{\mathbf{x}}^{\mathrm{rev}}\rangle_{\mathbf{p}}$ averaged over $\mathbf{p}$ plays an equivalent role, which is recognized as the nonequilibrium steady-state current in position space [8,33]. Interestingly, $\langle\mathbf{j}_{\mathbf{x}}^{\mathrm{rev}}\rangle_{\mathbf{p}} \neq \mathbf{0}$ is not a necessary condition for irreversibility, which will be observed from the second example later.

V. DETAILED BALANCE

Stochastic reversibility of a given process implies its time-reversal symmetry in the probabilistic sense, which defines *equilibrium*. This is also known to be enforced

by the detailed balance condition (DB) between all pairs of microscopic states [28,34]. However, when odd-parity variables are present, a special care is necessary to relate the DB condition to stochastic reversibility. The DB condition, in general, is given in terms of the transition rates between $\mathbf{q}$ and $\mathbf{q}'(\neq \mathbf{q})$ as

$$\langle \mathbf{q}'|\mathsf{L}|\mathbf{q}\rangle \rho(\mathbf{q}) = \langle \varepsilon\mathbf{q}|\mathsf{L}|\varepsilon\mathbf{q}'\rangle \rho(\varepsilon\mathbf{q}'). \tag{18}$$

Conventionally, the mirror symmetry of the steady-state PDF, $\rho_{ss}(\mathbf{q}) = \rho_{ss}(\varepsilon\mathbf{q})$, is also required for the system to be in equilibrium. Recently, for *discrete* state jumping dynamics [11], DB condition has been reported not necessarily to imply mirror symmetry and the two conditions have been reported to be, in fact, independent.

For the Brownian motion considered here, we analyze the DB equation, Eq. (18), without assuming the mirror symmetry beforehand. In terms of the differential operator $\mathsf{L}(\mathbf{q})$ given in Eq. (3), the DB condition is $\mathsf{L}(\mathbf{q}')\delta(\mathbf{q}' - \mathbf{q})\rho(\mathbf{q}) = \mathsf{L}(\varepsilon\mathbf{q})\delta(\mathbf{q} - \mathbf{q}')\rho(\varepsilon\mathbf{q}')$. Changing $\rho(\mathbf{q})$ to $\rho(\mathbf{q}')$ on the left-hand side and moving $\rho(\varepsilon\mathbf{q}')$ on the right-hand side to the front, we can rewrite this as

$$\mathsf{L}(\mathbf{q}')\rho(\mathbf{q}')\delta(\mathbf{q}' - \mathbf{q}) = \rho(\varepsilon\mathbf{q}')\mathsf{L}^\dagger(\varepsilon\mathbf{q}')\delta(\mathbf{q}' - \mathbf{q}), \tag{19}$$

where the operator $\mathsf{L}^\dagger$ is defined as

$$\mathsf{L}^\dagger(\mathbf{q}) \equiv -\frac{\mathbf{p}}{m} \cdot \partial_{\mathbf{x}} - \left(\mathbf{f} - \frac{\mathbf{p}}{m} \cdot \mathsf{G}^{\mathrm{t}} + \partial_{\mathbf{p}} \cdot \mathsf{D}\right) \cdot \partial_{\mathbf{p}}. \tag{20}$$

Equation (19) can now be regarded as the equality between differential operators acting on delta functions. From the part that does not involve any derivatives on the delta function, the steady-state condition $\mathsf{L}(\mathbf{q}')\rho(\mathbf{q}') = 0$ follows, so $\rho(\mathbf{q}') = \rho_{ss}(\mathbf{q}')$ and $\partial_{\mathbf{q}} \cdot \mathbf{j}_{ss,\mathbf{q}} = 0$. Comparing the terms proportional to $\partial_{\mathbf{x}'}\delta(\mathbf{q}' - \mathbf{q})$, we get the mirror symmetry

$$\rho_{ss}(\mathbf{q}) = \rho_{ss}(\varepsilon\mathbf{q}). \tag{21}$$

Thus, the mirror symmetry is a direct consequence of the DB condition for Brownian dynamics, in contrast to discrete state-jumping dynamics. For a more general continuous stochastic dynamics involving multiplicative noises, however, the situation is not so simple. In that case, the mirror symmetry follows from the DB only after assuming a certain condition for the multiplicative noise strengths. In general, the two conditions remain independent [35]. The broken mirror symmetry manifests the existence of a nonzero average current $\langle \mathbf{j}_{\mathbf{x}}^{\mathrm{rev}}\rangle_{\mathbf{p}}$ in position space, as seen from Eq. (15).

The terms proportional to $\partial_{\mathbf{p}'}\delta(\mathbf{q}' - \mathbf{q})$ give the vanishing irreversible steady-state current

$$\mathbf{j}_{ss,\mathbf{p}}^{\mathrm{ir}} = \left(\mathbf{f}^{\mathrm{ir}}(\mathbf{q}) - \mathsf{G} \cdot \frac{\mathbf{p}}{m} - \mathsf{D} \cdot \partial_{\mathbf{p}}\right)\rho_{ss}(\mathbf{q}) = \mathbf{0}. \tag{22}$$

This is consistent with the previous result in Eq. (17) in that the DB condition characterizing equilibrium guarantees stochastic reversibility even in the presence of momentum-dependent forces. With the broken DB, the total EP should increase incessantly in time in NEQ steady states. Higher-order derivative terms do not provide any additional condition.

VI. UNCONVENTIONAL EP

The definition of the environmental EP, Eqs. (10) and (11), in terms of the irreversibility measure should be checked against its conventional definition in thermodynamics. Without any momentum-dependence force, $\mathbf{f}(\mathbf{q}) = \mathbf{f}(\mathbf{x}) = \mathbf{f}^{\mathrm{rev}}$ and $\mathbf{f}^{\mathrm{ir}} = 0$, this definition has already been shown in the literature to be consistent with the conventional one [8,9]. For example, in the case of a single heat reservoir at temperature T with $\mathsf{D} = T\mathsf{G}$, Eq. (11) is simplified as

$$\frac{\mathrm{d}S_{env}}{\mathrm{d}t} = -\frac{1}{T}\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \mathbf{f}\right) \cdot \frac{\mathbf{p}}{m} = \frac{1}{T}\frac{\mathrm{d}Q}{\mathrm{d}t} , \tag{23}$$

where $\mathrm{d}Q$ is the heat flow into the reservoir (minus work done by the thermal forces) along a given trajectory during an infinitesimal time interval $\mathrm{d}t$. (Note that heat should be evaluated with the midpoint value of $\mathbf{p}/m$ in the above equation to maintain the energy conservation law [36]). Thus, the environmental EP is given solely by the conventional Clausius EP as $\mathrm{d}S_{env} = \mathrm{d}S_{res} = \mathrm{d}Q/T$. With general (multiple) reservoirs, we find

$$\frac{\mathrm{d}S_{env}}{\mathrm{d}t} = \frac{\mathrm{d}S_{res}}{\mathrm{d}t} = -\left(\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} - \mathbf{f}\right) \cdot \mathsf{D}^{-1} \cdot \mathsf{G} \cdot \frac{\mathbf{p}}{m} . \tag{24}$$

With the temperature matrix $\mathsf{T} = \mathsf{G}^{-1} \cdot \mathsf{D}$ and heat matrix differential $\mathrm{d}\mathsf{Q} = -(\mathrm{d}\mathbf{p} - \mathbf{f}\mathrm{d}t)(\mathbf{p}/m)^{\mathrm{t}}$, we get $\mathrm{d}S_{env} = \mathrm{Tr}(\mathsf{T}^{-1} \cdot \mathrm{d}\mathsf{Q}^{\mathrm{t}})$, which can be interpreted as the generalized Clausius EP into reservoirs.

With momentum-dependent forces, obviously the (generalized) Clausius EP rate is not identical to the environmental EP: Eq. (11) can be rewritten as $\mathrm{d}S_{env} = \mathrm{d}S_{res} + \mathrm{d}S_{uc}$, where the unconventional EP rate is simply given by the difference between Eqs. (11) and (24). The unconventional EP may be interpreted as an additional EP into an external agent that exerts momentum-dependent forces $\mathbf{f}(\mathbf{q}) = \mathbf{f}(\mathbf{x}, \mathbf{p})$. However, this external agent is not in the conventional form such as a thermal reservoir, so understanding this additional entropy in terms of the conventional thermodynamics is difficult. In fact, we do not know whether the unconventional EP has any feature of the conventional entropic measure, though evidently it contributes to the irreversibility in Eq. (6). Note that the unconventional EP is present even when $\mathbf{f}^{\mathrm{ir}} = 0$, if $\mathbf{f}^{\mathrm{rev}}$ includes a momentum dependence.

The average reservoir (generalized Clausius) EP rate can be obtained via algebra similar to that used in deriving Eq. (17) as

$$\left\langle \frac{\mathrm{d}S_{res}}{\mathrm{d}t}\right\rangle = \int \mathrm{d}\mathbf{q}\, \frac{\mathbf{p}}{m}\cdot\mathsf{G}^{\mathrm{t}}\cdot\left(\mathsf{D}^{-1}\cdot\mathsf{G}\cdot\frac{\mathbf{p}}{m} + \partial_{\mathbf{p}}\right)\rho(\mathbf{q}) ; \tag{25}$$

similarly, we find

$$\left\langle \frac{\mathrm{d}S_{uc}}{\mathrm{d}t}\right\rangle = \int \mathrm{d}\mathbf{q}\, \rho(\mathbf{q})\left[\mathbf{f}^{\mathrm{ir}}\cdot\mathsf{D}^{-1}\cdot\left(\mathbf{f}^{\mathrm{ir}} - 2\mathsf{G}\cdot\frac{\mathbf{p}}{m}\right)\right.$$
$$\left. -\partial_{\mathbf{p}}\cdot\left(\mathbf{f}^{\mathrm{rev}} - \mathbf{f}^{\mathrm{ir}}\right)\right]. \tag{26}$$

These two EP rates sum to the average environmental EP rate, which is consistent with the difference between Eqs. (17) and (16).

VII. EXAMPLES

We now demonstrate the novelty and the importance of the unconventional EP in a few examples that may be realized in experiments. First, we investigate a cold-damping problem with a dissipative external force $\mathbf{f}^{\mathrm{ir}} = -\mathsf{G}' \cdot \mathbf{p}/m$ with a positive-definite G'. For simplicity, we take $\mathbf{f}^{\mathrm{rev}} = 0$, $\mathsf{G} = \gamma\mathsf{I}$, and $\mathsf{D} = D\mathsf{I}$ in two dimensions, with the identity matrix I and the reservoir temperature $T = D/\gamma$. The steady-state PDF, $\rho_{ss}(\mathbf{q})$, can be easily obtained in any linear diffusion system, as shown in Refs. 18 and 33. For given $\mathsf{G}' = \gamma \begin{pmatrix} r & a \\ b & r \end{pmatrix}$ with a, $b > 0$ and $r^2 > ab$, straightforward algebra with Eqs. (17) and (25) yields

$$\left\langle \frac{\mathrm{d}S_{tot}}{\mathrm{d}t} \right\rangle = \frac{\gamma}{m} \frac{(a-b)^2}{2(1+r)} \tag{27}$$

$$\left\langle \frac{\mathrm{d}S_{res}}{\mathrm{d}t} \right\rangle = \frac{\gamma}{m} \left[\frac{4(1+r)^2 + (a-b)^2}{2(1+r)\{(1+r)^2 - ab\}} - 2 \right] .$$

As the average system EP vanishes in the steady state, $\langle \mathrm{d}S_{uc}/\mathrm{d}t \rangle$ is simply the difference between the above two EP rates. We can see explicitly that the reservoir EP rate (heat production rate) can be negative for large r, but the total EP rate remains positive (second law of thermodynamics) due to the unconventional EP contribution.

In a cold-damping problem, in the steady state, heat should flow into the system at a lower temperature from the higher-temperature reservoir, which dissipates due to an additional dissipative external force. Thus, the reservoir entropy decreases incessantly, which is compensated for and usually overridden by the unconventional EP to maintain the second law of thermodynamics. In a practical cold-damping problem, a feedback mechanism through measurement of the particle momentum operates continuously as this additional dissipative force; thus, the usual information entropy (mutual information) [16] enters into the total EP in addition, which will be discussed elsewhere [37].

As a second example, we consider the case with $\mathbf{f}^{\mathrm{ir}} = -\mathsf{A} \cdot \mathbf{p}/m$ with an antisymmetric $\mathsf{A} = -\mathsf{A}^{\mathrm{t}}$. For $\mathbf{f}^{\mathrm{rev}} = -\partial_{\mathbf{x}}V$ and $\mathsf{D} = T\mathsf{G}$, this can describe a charged particle's motion under a confined potential $V(\mathbf{x})$ with a magnetic field in contact with a single-temperature reservoir. From Eqs. (2) and (3), one can easily show that $\rho_{ss}(\mathbf{q}) \propto e^{-\frac{1}{T}\{\mathbf{p}^2/2m + V(\mathbf{x})\}}$, which is the same Boltzmann distribution as for the equilibrium case without $\mathbf{f}^{\mathrm{ir}}$. We find $\langle \mathrm{d}S_{res}/\mathrm{d}t \rangle = 0$ (no heat production because $\mathbf{f}^{\mathrm{ir}}$ does no work) from Eq. (25), but the DB is broken due to a nonzero irreversible current in the momentum

space as in Eq. (14), leading to a nonzero total EP in the steady state in Eq. (17):

$$\mathbf{j}^{\mathrm{ir}}_{ss,\mathbf{p}} = -\mathsf{A} \cdot \frac{\mathbf{p}}{m}\rho_{ss} , \left\langle \frac{\mathrm{d}S_{tot}}{\mathrm{d}t} \right\rangle = \frac{1}{m}\mathrm{Tr}\left[\mathsf{G}^{-1}\mathsf{A}\mathsf{A}^{\mathrm{t}}\right] . \tag{28}$$

Note that the unconventional EP is the only source for the total EP in this case.

For a charged particle's motion in two dimensions with an external uniform magnetic field $\mathbf{B}$ applied in the perpendicular direction, the antisymmetric matrix can be written as $\mathsf{A} = B \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then, we get $\langle \mathrm{d}S_{tot}/\mathrm{d}t \rangle = 2B^2/(\gamma m)$ with $\mathsf{G} = \gamma\mathsf{I}$. Due to the mirror symmetry in the steady state, there is no net current in position space, $\langle \mathbf{j}^{\mathrm{rev}}_{\mathbf{x}} \rangle_{\mathbf{p}} = \mathbf{0}$. For the system in a confined geometry, one can expect a boundary current, which represents irreversibility in position space, even though there is no net bulk current. Irreversibility in momentum space implies the existence of a residual force persistent in thermal fluctuation, which can be observed in this example from the helicity describing the tendency of circulation around the external magnetic field. The helicity can be measured by $\langle \mathbf{p}(t+\Delta t) \times \mathbf{p}(t) \rangle/(\Delta t)$ for $\Delta t \to 0$ with the cross product $\times$, which can be found as $\int \mathrm{d}\mathbf{p}\, \mathbf{j}^{\mathrm{ir}}_{ss,\mathbf{p}} \times \mathbf{p} = 2T\mathbf{B}$. This signifies a difference from the equilibrium motion in the absence of a magnetic field. We remark that, in the case of an internally current-induced magnetic field which should change its sign under time reversal [38], the magnetic force becomes reversible; thus, $\mathbf{f}^{\mathrm{ir}} = 0$. Then, the DB is restored, and the total EP is zero in the steady state.

In summary, we have considered the EP in a system containing both even and odd variables under time reversal. We have obtained explicit expressions for the EP's and their average values. In the presence of an external momentum-dependent force, the environmental EP contains not only the usual reservoir EP due to heat transfer but also the unconventional EP. This additional EP is crucial for the validity of the second law of thermodynamics.

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